- LQR

$$\min_{x[0:N],\, u[0:N-1]} J = \sum_{n=0}^{N-1}\left[\tfrac{1}{2}x_n^T Q_n x_n + q_n^T x_n + \tfrac{1}{2}u_n^T R_n u_n + r_n^T u_n\right] + \tfrac{1}{2}x_N^T Q_f x_N + q_f^T x_N$$

$$\text{s.t.} \quad x_{n+1} = A_n x_n + B_n u_n \qquad n = 0 \cdots N-1$$

$$x_0 = x_{\text{init}}$$

- Can locally approximate many nonlinear problems
- Computationally tractable

- Riccati

let
$$J_n(x) = \min_{x[n:N],\, u[n:N-1]} \sum_{k=n}^{N-1}\left[\tfrac{1}{2}x_k^T Q_k x_k + q_k^T x_k + \tfrac{1}{2}u_k^T R_k u_k + r_k^T u_k\right] + \tfrac{1}{2}x_N^T Q_f x_N + q_f^T x_N$$

$$\text{s.t.} \quad x_{k+1} = A_k x_k + B_k u_k \qquad k = n \cdots N-1$$

$$x_n = x$$

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2$$

Claim: each $J_n(x)$ is quadratic in x $\quad J_n(x) = \tfrac{1}{2}x^T P_n x + q_n^T x + \gamma_n$

1° $J_N(x) = \tfrac{1}{2}x^T Q_f x + q_f^T x \quad \Rightarrow P_N = Q_f \quad p_N = q_f \quad \gamma_N = 0$

2° if $J_{n+1}(x) = \tfrac{1}{2}x^T P_{n+1} x + p_{n+1}^T x + \gamma_{n+1}$

$$J_n(x) = \min_u \Big[\tfrac{1}{2}x^T Q_n x + q_n^T x + \tfrac{1}{2}u^T R_n u + r_n^T u + \tfrac{1}{2}(A_n x + B_n u)^T P_{n+1}(A_n x + B_n u) + p_{n+1}^T(A_n x + B_n u) + \gamma_{n+1}\Big]$$

$$= \min_u \Big[\tfrac{1}{2}x^T(Q_n + A_n^T P_{n+1} A_n)x + \tfrac{1}{2}u^T(R_n + B_n^T P_{n+1} B_n)u + (q_n + A_n^T p_{n+1})^T x + (r_n + B_n^T p_{n+1})^T u + x^T A_n^T P_{n+1} B_n u\Big] + \gamma_{n+1}$$

$$\nabla_u = (R_n + B_n^T P_{n+1} B_n)u + (r_n + B_n^T p_{n+1}) + B_n^T P_{n+1} A_n x := 0$$

$$u^* = -\underbrace{(R_n + B_n^T P_{n+1} B_n)^{-1}(B_n^T P_{n+1} A_n)}_{k_n}x - \underbrace{(R_n + B_n^T P_{n+1} B_n)^{-1}(r_n + B_n^T p_{n+1})}_{d_n}$$

$$= \tfrac{1}{2}x^T(Q_n + A_n^T P_{n+1} A_n)x + \tfrac{1}{2}(k_n x + d_n)^T(R_n + B_n^T P_{n+1} B_n)(k_n x + d_n) + (q_n + A_n^T p_{n+1})^T x + (r_n + B_n^T p_{n+1})^T(-k_n x - d_n) + x^T A_n^T P_{n+1} B_n(-k_n x - d_n) + \gamma_{n+1}$$

note that k_n and P_n only depends on Q_n R_n A_n B_n

$$= \tfrac{1}{2}x^T\Big[\underbrace{Q_n + k_n^T R_n k_n + (A_n - B_n k_n)^T P_{n+1}(A_n - B_n k_n)}_{P_n}\Big]x + \Big[\underbrace{k_n^T(R_n d_n - r_n) + q_n + (A_n - B_n k_n)^T(p_{n+1} - P_{n+1} B_n d_n)}_{p_n}\Big]^T x + \underbrace{\tfrac{1}{2}d_n^T[R_n + B_n^T P_{n+1} B_n]d_n - (r_n + B_n^T p_{n+1})^T d_n + \gamma_{n+1}}_{\gamma_n}$$

$$k_n^T R_n k_n - (B_n k_n)^T P_{n+1}(A_n - B_n k_n) = k_n^T\left[R_n k_n - B_n^T P_{n+1}(A_n - B_n k_n)\right]$$

- MPC via OSQP

$$\min_{x[0:N]\ u[0:N-1]} J = \sum_{n=0}^{N-1}\left[\frac{1}{2}x_n^T Q_n x_n + q_n^T x_n + \frac{1}{2}u_n^T R_n u_n + r_n^T u_n\right] + \frac{1}{2}x_N^T Q_f x_N + q_f^T x_N$$

$$\text{St} \quad x_{n+1} = A_n x_n + B_n u_n \qquad n = 0 \cdots N-1$$

$$x_0 = x_{init}$$

$$x_n \in \mathcal{X} \qquad u_k \in \mathcal{U}$$

$$\Downarrow$$

$$\min_{x,u,\tilde{x}\,\tilde{u}} \quad J(x,u) + I(x_{n+1} = A_n x_n + B_n u_n) + I(\tilde{x}_n \in \mathcal{X}) + I(\tilde{u}_n \in \mathcal{U}) + I(x_0 = x_{init})$$

$$\text{St} \quad x_n = \tilde{x}_n \ : y_n \qquad n = 0 \cdots N$$

$$u_n = \tilde{u}_n \ : g_n \qquad n = 0, \cdots N-1$$

$$\begin{aligned} A = \ & J(x,u) + I(x_{n+1} = A_n x_n + B_n u_n) + I(\tilde{x}_n \in \mathcal{X}) + I(\tilde{u}_n \in \mathcal{U}) + I(x_0 = x_{init}) \\ & + \sum_{n=1}^{N} y_n^T (x_n - \tilde{x}_n) + \sum_{n=0}^{N-1} w_n^T (u_n - \tilde{u}_n) \\ & + \sum_{n=1}^{N} \frac{\rho}{2}\|x_n - \tilde{x}_n\|_2^2 + \sum_{n=0}^{N-1}\frac{\rho}{2}\|u_n - \tilde{u}_n\|_2^2 \\ = \ & J(x,u) + I(x_{n+1} = A_n x_n + B_n u_n) + I(\tilde{x}_n \in \mathcal{X}) + I(\tilde{u}_n \in \mathcal{U}) + I(x_0 = x_{init}) \\ & + \sum_{n=0}^{N}\frac{\rho}{2}\|x_n - \tilde{x}_n + \frac{1}{\rho}y_n\|_2^2 + \sum_{n=0}^{N-1}\frac{\rho}{2}\|u_n - \tilde{u}_n + \frac{1}{\rho}g_n\|_2^2 \\ & - \sum_{n=0}^{N}\frac{1}{2\rho}\|y_n\|_2^2 - \sum_{n=0}^{N-1}\frac{1}{2\rho}\|g_n\|_2^2 \end{aligned}$$

Step 1. $\min_{x\,u} A$

$$\Downarrow$$

$$\min_{x\,u} \sum_{n=0}^{N-1}\begin{bmatrix} \frac{1}{2}x_n^T(Q_n + \rho I)x_n + x_n^T(q_n + y_n - \rho\tilde{x}_n) \\ + \frac{1}{2}u_n^T(R_n + \rho I)u_n + u_n^T(r_n + g_n - \rho\tilde{u}_n) \end{bmatrix} + \frac{1}{2}x_N^T(Q_f + \rho I)x_N + x_N^T(q_f + y_N - \rho\tilde{x}_N)$$

$$\text{St} \quad x_{n+1} = A_n x_n + B_n u_n$$

$$x_0 = x_{init}$$

Solve by LQR

Step 2: $\min_{\tilde{x}\,\tilde{u}} A$

$$\Downarrow$$

$$\min_{\tilde{x}\,\tilde{u}} \sum_{n=0}^{N}\frac{\rho}{2}\|x_n - \tilde{x}_n + \frac{1}{\rho}y_n\|_2^2 + \sum_{n=0}^{N-1}\frac{\rho}{2}\|u_n - \tilde{u}_n + \frac{1}{\rho}g_n\|_2^2$$

$$\text{St} \quad \tilde{x}_n \in \mathcal{X} \quad \tilde{u}_n \in \mathcal{U}$$

$$\Downarrow$$

$$\tilde{x}_n = \Pi_{\mathcal{X}}(x_n + \frac{1}{\rho}y_n) \qquad \tilde{u}_n = \Pi_{\mathcal{U}}(u_n + \frac{1}{\rho}g_n)$$

Step 3:

$$y_n := y_n + \rho(x_n - \tilde{x}_n)$$

$$g_n := g_n + \rho(u_n - \tilde{u}_n)$$