$$\min_x \frac{1}{2}x^TPx + q^Tx$$

$$\text{st} \quad Ax \in C$$

$$\Downarrow$$

$$\min_{x,z} \frac{1}{2}x^TPx + q^Tx$$

$$\text{st} \quad Ax = z \quad : y$$

$$z \in C$$

$$\mathcal{L}(x,z,y) = \frac{1}{2}x^TPx + q^Tx + y^T(Ax-z)$$
$$= \frac{1}{2}x^TPx + q^Tx + (A^Ty)^Tx - z^Ty$$

$$g(y) = \inf_x \left\{\frac{1}{2}x^TPx + q^Tx + (A^Ty)^Tx\right\} + \inf_{z\in C}\{-z^Ty\}$$

$$g(y) = \inf_x \left\{\frac{1}{2}x^TPx + q^Tx + (A^Ty)^Tx\right\} + \inf_{z\in C}\{-z^Ty\}$$
$$\le \inf_{\substack{Ax=z\\ z\in C}} \left\{\frac{1}{2}x^TPx + q^Tx + y^T(Ax-z)\right\}$$
$$= p^*$$

KKT condition

$$Ax^* = z^*$$
$$z^* \in C$$
$$Px^* + q + A^Ty^* = 0$$
$$z^* = \arg\max_z \{z^Ty^*\} \iff y^* \in N_C(z^*) = \{g : g^T(u-z^*) \le 0 \quad \forall u \in C\}$$

Theorem : (Strong Alternative)

let $P = \{x : Ax \in C\}$

$D = \{y : A^Ty = 0 \quad \sup_{u\in C} u^Ty < 0\}$

if $D \neq \phi$ then $P = \phi$

if $D \neq \phi$ $\exists \tilde{y}$ st $A^T\tilde{y} = 0$ $\sup_{u\in C} u^T\tilde{y} < 0$

then $\inf_x (Ax)^T\tilde{y} = \inf_x x^T(A^T\tilde{y}) = 0$

$$\Downarrow$$

$$\begin{cases} \inf_x (Ax)^T\tilde{y} = 0 \\ \sup_{u\in C} u^T\tilde{y} < 0 \end{cases}$$

$\{Ax : x\} \cap C = \phi$

since $\{z : z^T\tilde{y} = 0\}$ is a separating hyperplane

OSQP

$$\min_{x,z} \ \frac{1}{2}x^T P x + q^T x$$
$$\text{s.t.} \quad Ax = z$$
$$z \in C$$

$$\min_{x,z} \ \frac{1}{2}x^T P x + q^T x + I(Ax=z) + I(z \in C)$$

$$\min_{x,z,\tilde{x},\tilde{z}} \ \frac{1}{2}\tilde{x}^T P \tilde{x} + q^T \tilde{x} + I(A\tilde{x}=\tilde{z}) + I(z \in C)$$
$$\text{s.t.} \quad \tilde{x} = x \ : w$$
$$\tilde{z} = z \ : y$$

this constraint looks redundant, but it makes the KKT matrix invertible

$$\begin{aligned} \mathcal{L}(x,z,\tilde{x},\tilde{z},w,y) &= \frac{1}{2}\tilde{x}^T P \tilde{x} + q^T\tilde{x} + I(A\tilde{x}=\tilde{z}) + I(z\in C) \\ &\quad + w^T(\tilde{x}-x) + y^T(\tilde{z}-z) \\ &\quad + \frac{\sigma}{2}\|\tilde{x}-x\|_2^2 + \frac{\rho}{2}\|\tilde{z}-z\|_2^2 \\ &= \frac{1}{2}\tilde{x}^T P \tilde{x} + q^T\tilde{x} + I(A\tilde{x}=\tilde{z}) + I(z\in C) \\ &\quad + \frac{\sigma}{2}\|\tilde{x}-x+\frac{1}{\sigma}w\|_2^2 + \frac{\rho}{2}\|\tilde{z}-z+\frac{1}{\rho}y\|_2^2 \\ &\quad - \frac{1}{2\sigma}\|w\|_2^2 - \frac{1}{2\rho}\|y\|_2^2 \end{aligned}$$

$$\min_{\tilde{x},\tilde{z}} \ \frac{1}{2}\tilde{x}^T P \tilde{x} + q^T\tilde{x} + \frac{\sigma}{2}\|\tilde{x}-x+\frac{1}{\sigma}w\|_2^2 + \frac{\rho}{2}\|\tilde{z}-z+\frac{1}{\rho}y\|_2^2$$
$$\text{s.t.} \quad A\tilde{x} = \tilde{z} \quad : v$$

$$\begin{cases} A\tilde{x} = \tilde{z} \\ \nabla_{\tilde{x}}\mathcal{L} = P\tilde{x} + q + \sigma(\tilde{x}-x+\frac{1}{\sigma}w) + A^T v = 0 \\ \nabla_{\tilde{z}}\mathcal{L} = \rho(\tilde{z}-z+\frac{1}{\rho}y) - v = 0 \quad \Rightarrow \tilde{z} = z - \frac{1}{\rho}y + \frac{1}{\rho}v \end{cases}$$

$$\begin{bmatrix} P+\sigma I & A^T \\ A & -\frac{1}{\rho}I \end{bmatrix} \begin{bmatrix} \tilde{x} \\ v \end{bmatrix} = \begin{bmatrix} -q+\sigma x - w \\ z - \frac{1}{\rho}y \end{bmatrix}$$

$$\min_{x,z} \ I(z\in C) + \frac{\sigma}{2}\|\tilde{x}-x+\frac{1}{\sigma}w\|_2^2 + \frac{\rho}{2}\|\tilde{z}-z+\frac{1}{\rho}y\|_2^2$$

$$\nabla_x = \sigma(\tilde{x}-x+\frac{1}{\sigma}w) := 0 \qquad x = \tilde{x} + \frac{1}{\sigma}w$$
$$z = \Pi_C(\tilde{z} + \frac{1}{\rho}y)$$

$$\min_{x,z,\tilde{z}} \ \frac{1}{2}x^T P x + q^T x + I(Ax=\tilde{z}) + I_C(z)$$
$$\text{s.t.} \quad \tilde{z} = z \ : y$$

$$\begin{aligned} \mathcal{L}(x,z,\tilde{z},y) &= \frac{1}{2}x^T P x + q^T x + I(Ax=\tilde{z}) + I_C(z) \\ &\quad + y^T(\tilde{z}-z) + \frac{\rho}{2}\|\tilde{z}-z\|_2^2 \\ &= \frac{1}{2}x^T P x + q^T x + I(Ax=\tilde{z}) + I_C(z) \\ &\quad + \frac{\rho}{2}\|\tilde{z}-z+\frac{1}{\rho}y\|_2^2 - \frac{1}{2\rho}\|y\|_2^2 \end{aligned}$$

$$\min_{x,\tilde{z}} \ \frac{1}{2}x^T P x + q^T x + \frac{\rho}{2}\|\tilde{z}-z+\frac{1}{\rho}y\|_2^2$$
$$\text{s.t.} \quad Ax = \tilde{z}$$

$$\begin{cases} Ax = \tilde{z} \\ \nabla_x\mathcal{L} = Px + q + A^T v = 0 \\ \nabla_{\tilde{z}}\mathcal{L} = \rho(\tilde{z}-z+\frac{1}{\rho}y) - v = 0 \quad \Rightarrow \tilde{z} = z - \frac{1}{\rho}y + \frac{1}{\rho}v \end{cases}$$

$$\underbrace{\begin{bmatrix} P & A^T \\ A & -\frac{1}{\rho}I \end{bmatrix}}_{\text{might not be full rank}} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -q \\ z - \frac{1}{\rho}y \end{bmatrix}$$

Given parameter $\rho>0$ $\sigma>0$ $\alpha\in(0,2)$ $\qquad$ $\alpha=1.6$ in OSQP paper

Initialize x^0 z^0 w^0 y^0

loop {

① $$\begin{bmatrix} P+\sigma I & A^T \\ A & -\frac{1}{\rho}I \end{bmatrix}\begin{bmatrix} \tilde{x} \\ v \end{bmatrix} = \begin{bmatrix} -q+\sigma x^k - w^k \\ z^k - \frac{1}{\rho}y^k \end{bmatrix}$$

$$\tilde{z} = z^k - \frac{1}{\rho}y^k + \frac{1}{\rho}v$$

② $x^{k+1} = \alpha\tilde{x} + (1-\alpha)x^k + \frac{1}{\sigma}w^k$ $\qquad$ $\tilde{x} + \frac{1}{\sigma}w^k$ when $\alpha=1$

③ $z^{k+1} = \Pi_C\left(\alpha\tilde{z} + (1-\alpha)z^k + \frac{1}{\rho}y^k\right)$

④ $w^{k+1} = w^k + \sigma\left(\alpha\tilde{x} + (1-\alpha)x^k - x^{k+1}\right)$ $\qquad$ remark: w is always 0

⑤ $y^{k+1} = y^k + \rho\left(\alpha\tilde{z} + (1-\alpha)^k - z^{k+1}\right)$

Check residual $\begin{cases} Ax = z \\ Px + q + A^Ty = 0 \end{cases}$

}

KKT condition $\begin{cases} Ax^* = z^* \\ z^* \in C \\ Px^* + q + A^Ty^* = 0 \\ z^* = \arg\max_{z\in C}\{z^Ty^*\} \end{cases}$

$z^* \in C \Rightarrow$ always satisfied by $z^{(k+1)} = \Pi_C\left(\alpha\tilde{z} + (1-\alpha)z^k + \frac{1}{\rho}y^k\right)$

$\Downarrow$

$$y^{k+1} = y^k + \rho\left(\alpha\tilde{z} + (1-\alpha)z^k - z^{k+1}\right)$$

$$= y^k + \rho\left[\alpha\tilde{z} + (1-\alpha)z^k - \Pi_C\left(\alpha\tilde{z} + (1-\alpha)z^k + \frac{1}{\rho}y^k\right)\right]$$

let $h = \alpha\tilde{z} + (1-\alpha)z^k + \frac{1}{\rho}y^k$

then $z^{k+1} = \Pi_C(h)$

$y^{k+1} = \rho\left(h - \Pi_C(h)\right)$

then $z^{k+1} = \arg\max_z\{z^Ty^{k+1}\}$