Polynomial Non-negativity

let $f(x)$ be a polynomial function of x.

problem : $\exists\, x$ s.t. $f(x)<0$?

1° No: f is called PSD. need proof or certificate

2° Yes: give an x s.t. $f(x)<0$

Sum-of-Square

let f be a polynomial,

f is non-negative $\underset{\nRightarrow}{\Leftarrow}$ $\exists$ polynomial $g_1 \cdots g_m$ s.t. $f = g_1^2 + \cdots + g_m^2$ → Sum-of-Square decomposition

In general non-negativity does not imply SoS

We can write any polynomial as a quadratic function of monomials

Eg. $f(x,y) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$

$$= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -\lambda \\ 2 & -7+2\lambda & -1 \\ -\lambda & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}$$

$$= b(x,y)^T Q(\lambda)\, b(x,y)$$

With $\lambda = 6$

$$Q(\lambda) = \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$ is PSD

$$f(x,y) = \left\| \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \right\|_2^2$$

$$= (\underbrace{2xy + y^2}_{g_1})^2 + (\underbrace{2x^2 + xy - 3y^2}_{g_1})^2$$

Suppose f = polynomial $(x_1, \cdots x_n)$ of degree $2d$

let $b(x_1, \cdots, x_n)$ be a vector of all monomials with degree $\le d$

f is SoS $\Leftrightarrow$ $\exists\, Q \succeq 0$ s.t. $\underbrace{f(x_1 \cdots x_n) = b(x,y)^T Q\, b(x,y)}_{\text{affine constraint on } Q \text{ by matching coefs}}$

if $Q \succeq 0$ and $f(x_1 \cdots x_n) = b(x_1 \cdots x_n)^T Q\, b(x_1, x_n)$

Can reconstruct the sum-of-square by

$$b(x,y)^T Q\, b(x,y) = b(x,y)^T U^T U\, b(x,y) = \|U b(x,y)\|_2^2 \qquad Q = U^T U \text{ is the cholesky factorization}$$

- Polynomial Function Global Optimization

let $F(x,y)$ be a polynomial

$$\min_{x,y} F(x,y)$$

$$\Downarrow$$

$$\max_{\gamma} \gamma$$

$$\text{s.t. } F(x,y) \geq \gamma \quad \forall x,y$$

$$\Downarrow$$

$$\max_{\gamma} \gamma$$

$$\text{s.t. } F(x,y) - \gamma \in SOS$$

$$\Downarrow$$

$$\left.\begin{aligned} &\max_{\gamma} \gamma \\ &\text{s.t. } [1 \ \ b(x)] \underbrace{\begin{bmatrix} -\gamma & \cdots \\ \vdots & \ddots \end{bmatrix}}_{Q} \begin{bmatrix} 1 \\ b(x) \end{bmatrix} = f(x) \\ &\underbrace{\begin{bmatrix} -\gamma & \cdots \\ \vdots & \ddots \end{bmatrix}}_{Q} \succeq 0 \end{aligned}\right\} \text{SDP}$$

remark:

$Q \succeq 0 \Rightarrow \gamma \leq 0$

and $\min_x f(x) \leq 0$ since $f(0) = 0$

(if f contains constant term, just drop it)

- S-procedure

consider $\begin{cases} \text{a scalar polynomial } p(x) \\ \text{a vector of polynomial } g(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{bmatrix} \end{cases}$

1° if $\exists$ polynomial multiplier $\lambda(x) = \begin{bmatrix} \lambda_1(x) \\ \vdots \\ \lambda_m(x) \end{bmatrix}$ s.t.

$$\begin{cases} p(x) + \lambda(x)^T g(x) \in SOS \\ \lambda_i(x) \in SOS \quad \forall i = 1, \cdots m \end{cases}$$

then $g(x) \leq 0 \Rightarrow p(x) \geq 0$

proof:

Consider the optimization problem

$$\min_x p(x)$$

$$\text{s.t. } g_i(x) \leq 0 \qquad \text{dual polynomial } \lambda_i(x)$$

$$\mathcal{L}(x,\lambda) = p(x) + \lambda(x)^T g(x) \in SOS$$

thus $\mathcal{L}(x,\lambda) \geq 0 \ \forall x$

thus $0 \leq \inf_x \mathcal{L}(x,\lambda)$

$$\leq \inf_{g_i(x) \leq 0} p(x) + \underbrace{\sum_i \lambda_i(x) \cdot g_i(x)}_{\leq 0}$$

$$\leq \inf_{g_i(x) \leq 0} p(x)$$

Same proof as for weak duality

2° if $\exists$ polynomial multiplier $\lambda(x) = \begin{bmatrix} \lambda_1(x) \\ \vdots \\ \lambda_m(x) \end{bmatrix}$ s.t.

$$p(x) + \lambda(x)^T g(x) \in SOS$$

then $g(x) = 0 \Rightarrow p(x) \geq 0$