**Lemma :**

let $P \in S^n$, $Q \in S^n$,

if $tr(P) \geq 0$, $tr(Q) < 0$, then $\exists x \in \mathbb{R}^n$ s.t. $x^T P x \geq 0$, $x^T Q x < 0$

let $Q = U \Lambda U^T$, $tr(Q) = tr(\Lambda) < 0$

let $v$ be a random vector, $P(v_i = 1) = P(v_i = -1) = 0.5$, $v_i \perp v_j$

$$(Uy)^T Q (Uy) = y^T U^T U \Lambda U^T U y = y^T \Lambda y = \sum_i y_i^2 \lambda_i = tr(\Lambda) < 0$$

$$(Uy)^T P (Uy) = y^T U^T P U y$$

$$E[y^T U^T P U y] = E\left[\sum_{ij} y_i y_j (U^T P U)_{ij}\right] = Tr(U^T P U) = tr(P) \geq 0$$

thus $\exists y \in \{-1, 1\}^n$ s.t. $y^T U^T P U y \geq 0$

can take $x = Uy$, $x^T P x \geq 0$, $x^T Q x < 0$

**Theorem: ( Homogeneous S-Lemma )**

Suppose $A, B \in S^n$, $\exists x$ s.t. $x^T A x > 0$,

then the following 2 statements are equivalent

$$\begin{cases} x^T A x \geq 0 \Rightarrow x^T B x \geq 0 \\ \exists \lambda \geq 0 \text{ s.t. } B \succeq \lambda A \end{cases}$$

$(\Uparrow)$ $x^T B x \geq \lambda x^T A x \geq 0$

$(\Downarrow)$ Consider the optimization problem,

(1) $$\min_x \ x^T B x \quad \text{s.t. } x^T A x \geq 0 \quad \Rightarrow p_1^*$$

we have $x^T A x \geq 0 \Rightarrow x^T B x \geq 0$ thus $p^* = 0$

Consider the optimization problem

(2) $$\min_x \ x^T B x \quad \text{s.t. } x^T A x \geq 0,\ x^T x = n \quad \Rightarrow p_2^*$$

let $\hat{x}$ be s.t. $\hat{x}^T A \hat{x} > 0 \Rightarrow \hat{x} \neq 0$

let $\tilde{x}$ be a rescale of $\hat{x}$ s.t. $\tilde{x}^T \tilde{x} = n$, and $\tilde{x}^T A \tilde{x} > 0$

thus $\tilde{x}^T B \tilde{x} \geq 0$. $p_2^* \geq p_1^* \geq 0$ and $p_2^* \neq +\infty$ since (2) is feasible the optimization problems are equivalent

(2) $\min_{x} \; x^TBx$ s.t. $x^TAx \geq 0$, $x^Tx = n$

(3) $\min_{X} \; \langle B, X \rangle$ s.t. $\langle A, X \rangle \geq 0$, $tr(X) = n$, $X \succeq 0$, $rank(X) = 1$

$\forall x$ feasible for (2). $X = xx^T$ is feasible for (3). and achieves same obj value
$\forall X$ feasible for (3) $X$ is rank 1, $X = xx^T$, $x$ is feasible for (2) and achieves same value
thus $p_2^* = p_3^* \geq 0$

relax (3) by dropping the rank constraints and get (4)

$$
(4) \quad \min_{X} \; \langle B, X \rangle \quad \text{s.t.} \quad \langle A, X \rangle \geq 0 \; : \mu, \quad tr(X) = n \; : \nu, \quad X \succeq 0 \; : \Lambda
$$

take the dual of (4) and get (5)

$$
L(X, \Lambda, \mu, \nu) = \langle B, X \rangle - \langle \Lambda, X \rangle - \mu \langle A, X \rangle + \nu (tr(X) - n)
$$

$$
= \langle B - \Lambda - \mu A + \nu I, X \rangle - n\nu
$$

$$
g(\Lambda, \mu, \nu) = -n\nu
$$

$$
\max_{\Lambda, \mu, \nu} \; -n\nu \quad \text{s.t.} \quad B - \Lambda - \mu A + \nu I = 0, \; \Lambda \succeq 0, \; \mu \geq 0 \quad \Rightarrow \quad \max_{\mu, \nu} \; -n\nu \quad \text{s.t.} \quad B - \mu A + \nu I \succeq 0, \; \mu \geq 0 \quad (5)
$$

(4) is strictly feasible
(5) is strictly feasible. (fix $\mu$, make $\nu$ large enough)
$p_4^* \geq 0$

let $X^*$ be the solution to 4, $X^* = DD^T$
$\langle A, X^* \rangle = tr(ADD^T) = tr(D^TAD) \geq 0$

$$\mathrm{tr}(BX^*) = \mathrm{tr}(D^T B D) = p_4^*$$

if $p_4^* < 0$, then $\mathrm{tr}(D^T B D) < 0$, $\mathrm{tr}(D^T A D) \geq 0$

then by Lemma 1, $\exists x$ s.t. $x^T D^T A D x \geq 0$, $x^T D^T B D x < 0$

contradict the assumption

thus $p_5^* = p_4^* \geq 0$

thus $\nu^* \leq 0$

thus $\exists\ \mu^* \geq 0$ s.t. $B - \mu^* A + \nu^* I \succeq 0$

thus $\mu^* \geq 0$, $B - \mu^* A \succeq 0$