- Notation

$x \in \mathbb{R}^n$ or $\mathbb{R}^{m \times n}$ $\quad x \geq 0$ if x is element-wise nonneg

- Positivity

$\left[\begin{array}{l} \text{if } A \geq 0 \text{ and } z \geq 0, \text{ then } Az \geq 0 \\ \text{if } Az \geq 0 \ \forall\, z \geq 0, \text{ then } A \geq 0 \end{array}\right]$ matrix multiplication preserves non-negativity if and only if the matrix is non-neg

$\left[\begin{array}{l} \text{if } A > 0 \text{ and } z \geq 0\ z \neq 0, \text{ then } Az > 0 \\ \text{if } Az > 0 \ \forall\, z \geq 0, z \neq 0, \text{ then } A > 0 \end{array}\right.$

- Reducible Matrix and Primitive Matrix

Definition: Reducible

non-neg square T is reducible if

$\exists$ permutation matrix P s.t. $P^T T P = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ $\quad X\ Z$ square

A is irreducible if it's not reducible

Theorem:

non-negative square matrix T is irreducible if

$\forall\, i,j = 1 \cdots n \quad \exists\, k(i,j) \text{ s.t. } T_{ij}^{k(ij)} > 0$

$\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$

$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} AX & AY+BZ \\ 0 & CZ \end{bmatrix}$

if A reducible. $P^T T P = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ $\quad T = P \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} P^T$

$\forall\, k, \ T^k = \left(P \begin{bmatrix} Y & Y \\ 0 & Z \end{bmatrix} P^T\right)^k = P \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}^k P^T$ has 0 entry

$\therefore \exists\, ij \ \forall\, k \in \mathbb{N} \text{ s.t. } T_{ij}^k = 0$

Definition: Primitive

non-neg matrix T is primitive if

$\exists\, k \in \mathbb{N} \text{ s.t. } A^k > 0$

Theorem:

primitive $\Rightarrow$ irreducible $\quad \exists \forall \Rightarrow \forall \exists$

Theorem

let T be a non-neg square matrix

if T is irreducible, then $(I+A)$ is primitive

T is irreducible, $\forall\, ij\ \exists\, k(ij)$ s.t. $T_{ij}^{k(ij)} > 0$

let $k = \max_{ij} k(ij)$

$$(T+I)^k_{ij} = \sum_{n=1}^{k} \binom{k}{n} T^n_{ij} > 0$$

• Perron – Frobenius Theorem

Theorem: (Perron - Frobenius)

Let T be irreducible (which implies non-neg & square)

then

1° T has a positive real eigenvalue λ_{pf}, $|\lambda| \le \lambda_{pf}$ $\forall$ eigenvalue λ

2° The associated eigenvector x of λ_{pf} is positive

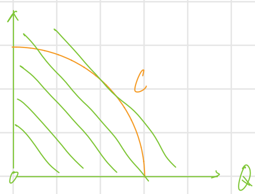
3° λ_{pf} is unique (has algebraic and geometric multiplicity 1)

4° Any non-negative eigenvector is a multiply of x

1°

let $Q = \{x \in \mathbb{R}^n \mid x \ge 0,\ x \ne 0\}$

$C = \{x \ge 0 \mid \|x\|_2 = 1\}$



T is irreducible, $I+T$ is primitive $\quad X = (I+T)^k > 0$

$\forall\, z \in Q$, let $L(z) = \max\{s : sz \le Tz\} = \min_{1 \le i \le n,\ z_i \ne 0} \frac{(Tz)_i}{z_i}$

$L(rz) = L(z)\quad \forall\, r > 0$

$\left.\begin{array}{l} z \le y \quad z \ne y \Rightarrow Xz < Xy \\ XT = TX \end{array}\right]$ if $sz \le Tz$ then $sXz \le XTz = TXz$

$\Downarrow$

$L(Xz) \ge L(z)$ ①

If $L(z)z \ne Tz$, and we have $L(z)z \le Tz$ by definition of $L(z)$

then $L(z)Xz < XTz = TXz$ ②

combining ① & ②

when $L(Xz) = L(z)$, we have

$L(z)z \neq Tz \Rightarrow L(Xz)Xz = L(z)Xz < TXz$

$\Downarrow$

Contradict with the definition of L

thus $L(z)z = Tz$

z is an eigenvector of T with eigenvalue $L(z)$

$L(Xz) \geq L(z)$

if $L(Xz) = L(z)$, then z is a eigenvector of T with eigenvalue $L(z)$

2° Find a positive eigenvector

we have $L(Xz/\|Xz\|_2) = L(Xz) \geq L(z) = L(z/\|z\|_2)$

thus $L(Q) = L(C)$

L is continuous on C & C is compact

L attains maximum and minimum over C

suppose L attains maximum at $L(u/\|u\|_2) = L(u)$

$L(Xu/\|Xu\|_2) = L(Xu) \geq L(u) = L(u/\|u\|_2)$

thus $L(Xu) = L(u)$

u is a eigenvector with $L(u)$ as it's eigenvalue

$(I+T)^k u = (I+L(u))^k u$

$(I+T)^k > 0 \quad u \geq 0 \quad u \neq 0 \quad \Rightarrow (I+L(u))^k u > 0 \quad \Rightarrow u > 0$

$T \geq 0$ is irreducible, T cannot have all-zero row, $Tu > 0$, $L(u) = \max\{su \leq Tu\} > 0$

3° Showing that $L(u^*)$ is the maximum eigenvalue

let y be any eigenvector with eigenvalue λ, (λ y may be complex)

$Ty = \lambda y$

$\lambda y_i = \sum_j T_{ij} y_j \qquad |\lambda||y_i| = |\lambda y_i| = |\sum_j T_{ij} y_j| \leq \sum_j T_{ij}|y_j|$

$|\lambda||y| \leq T|y|$

Thus $|\lambda| \leq L(|y|) \leq L(u^*)$

4°

$$\lambda_{pf}(|A|) = \inf_{S} \|S^{-1} A S\|_\infty$$

s.t. $S = \mathrm{diag}(s)$ $\quad s > 0$

$$\|A\|_\infty = \max_i \sum_j |A_{ij}|$$

$$\|S^{-1}AS\|_\infty = \max_i \sum_j |S^{-1}AS|_{ij}$$

$$= \max_i \sum_j 1/s_i \, |AS|_{ij}$$

$$= \max_i 1/s_i \sum_j |A_{ij} s_j|$$

$$= \max_i 1/s_i \sum_j |A_{ij}| s_j$$

$$= \max_i (|A|s)_i / s_i$$

$$? \left\{ \begin{aligned} &\inf_s \|S^{-1}AS\|_\infty = \inf_s \max_i \frac{(|A|s)_i}{s_i} \\ &\lambda_{pf}(|A|) = \max_{s \geq 0,\, s \neq 0} \ \min_{i,\, s_i \neq 0} \frac{(|A|s)_i}{s_i} \end{aligned} \right.$$