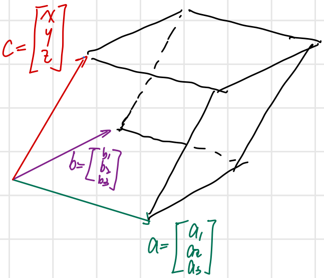
- Cross product

$$\text{volume}(\text{parallelepiped}) = f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

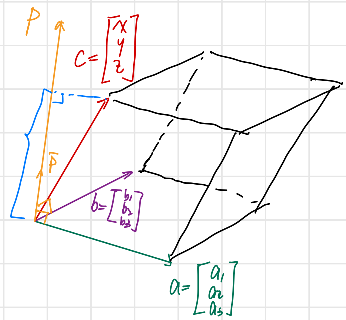


$$= \left| \det \begin{bmatrix} x & a_1 & b_1 \\ y & a_2 & b_2 \\ z & a_3 & b_3 \end{bmatrix} \right|$$

$$= \left| x(a_2 b_3 - a_3 b_2) + y(a_3 b_1 - a_1 b_3) + z(a_1 b_2 - a_2 b_1) \right|$$

f is a linear function (if we ignore the abs) in $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

thus it can be represented by inner product



let $\bar{p}$ be a unit vector s.t. $p \perp a$, $p \perp b$

height of the polyhedron

then $\text{volume}(\text{parallelepiped}) = \left| \text{area}(\text{parallelogram } a, b) \cdot c^T \bar{p} \right|$

$$= \left| c^T \left(\bar{p} \cdot \text{area}(\square) \right) \right|$$

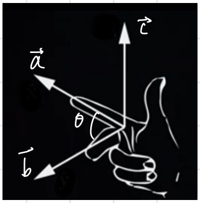
$$= \left| c^T p \right|$$

$$= \left| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \right|$$

p is a vector orthogonal to a & b,

and $\|p\| = \text{area}(\text{parallelogram } a, b)$

$$\left.\begin{aligned} p_1 &= a_2 b_3 - a_3 b_2 \\ p_2 &= a_3 b_1 - a_1 b_3 \\ p_3 &= a_1 b_2 - a_2 b_1 \end{aligned}\right\} \quad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \det \begin{bmatrix} i & a_1 & b_1 \\ j & a_2 & b_2 \\ k & a_3 & b_3 \end{bmatrix}$$



Cross product: $\vec{a} \times \vec{b} = \vec{c}$ where $\vec{a}$ $\vec{b}$ are independent

$\times : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3$

properties of $\vec{c} = \vec{a} \times \vec{b}$
$$\begin{cases} c \perp a \quad c \perp b \\ \|c\|_2 = \|a\|_2 \|b\|_2 \sin\theta = \text{area of parallelogram } a, b = \det\left(\begin{bmatrix} a & b \end{bmatrix}\right) \\ \text{direction of } c \text{ follows right-hand rule} \end{cases}$$

Cross product only defined for $\mathbb{R}^3$

In $\mathbb{R}^2$, no vector is orthogonal to a & b

In $\mathbb{R}^4$, infinite number of vecs are orthogonal to a & b

$c = a \times b$ is computed as $\begin{cases} c_1 = a_2 b_3 - a_3 b_2 \\ c_2 = a_3 b_1 - a_1 b_3 \\ c_3 = a_1 b_2 - a_2 b_1 \end{cases}$,

can be writen as $c = \begin{cases} [a]_\times b = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} | \\ b \\ | \end{bmatrix} \\ [b]_\times^T a = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix} \begin{bmatrix} | \\ a \\ | \end{bmatrix} \end{cases}$

Matrix Exponential & Rotation

Theorems $\begin{cases} e^{-A} = (e^A)^{-1} \\ e^A e^B = e^{A+B} \text{ if } AB = BA \\ e^{-A} \cdot e^A = I \end{cases}$

Theorem: if A is skew-symmetric, e^{tA} is orthonormal

e^A is orthonormal $\Leftrightarrow (e^A)^{-1} = (e^A)^T \Leftrightarrow e^{-A} = e^{A^T} \Leftarrow -A = A^T$

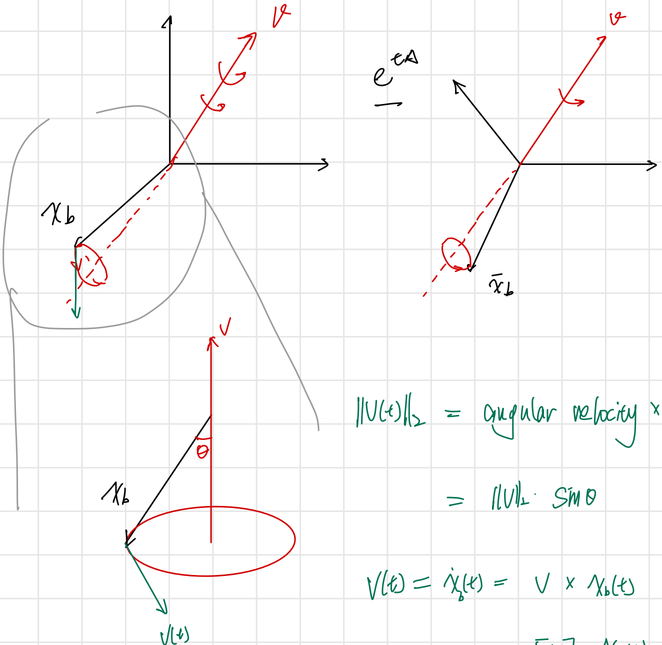
Theorem: if $A = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$, e^{tA} is a rotation matrix about $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, angular velocity is $\|v\|$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & -z & y \\ z & -\lambda & -x \\ -y & x & -\lambda \end{bmatrix}\right) = -\lambda^3 - xyz + xyz - \lambda y^2 - \lambda x^2 - \lambda z^2$$
$$= -\lambda\left(\lambda^2 + (x^2 + y^2 + z^2)\right)$$
$$= 0$$

$\lambda = \begin{cases} 0 \\ \pm i \cdot \|v\|_2 \end{cases}$ Also $\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

e^{tA} has $\begin{cases} \text{eigenvalue } e^{t\cdot 0} = 1 \text{ with eigenvector } v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \text{eigenvalue } e^{\pm i \|v\|_2 t} = \cos(\|v\|_2 t) + i\sin\theta(\|v\|_2 t) \end{cases}$

Since e^{tA} is orthonormal, it's a rotation matrix
$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ must be the rotation axis



$\|v(t)\|_2$ = angular velocity × radius

$= \|v\|_2 \cdot \sin\theta$

$v(t) = \dot{x}_b(t) = v \times x_b(t)$

$= [v]_\times x_b(t)$

$\dot{x}_b(t) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} x_b(t)$

Similarly $\dot{x}_b(t) = [v]_\times x_b(t)$

$\dot{y}_b(t) = [v]_\times y_b(t)$

$\dot{z}_b(t) = [v]_\times z_b(t)$

let $V_b = V$ be the description of v in original frame $\begin{bmatrix} | & | & | \\ x_b & y_b & z_b \\ | & | & | \end{bmatrix}$

$V_s(t)$ be the description of v in rotated frame $\begin{bmatrix} | & | & | \\ x_b(t) & y_b(t) & z_b(t) \\ | & | & | \end{bmatrix}$

$${}^b_sR(t) = \begin{bmatrix} | & | & | \\ x_b(t) & y_b(t) & z_b(t) \\ | & | & | \end{bmatrix}$$

$$V_b = {}^b_sR(t)\, V_s(t)$$

$$V_s(t) = {}^b_sR(t)^{-1}\, V_b = {}^b_sR(t)^T\, V_b$$

$${}^b_s\dot{R}(t) = [V]_\times\, {}^b_sR(t)$$

$$\dot{V}_s(t) = {}^b_s\dot{R}(t)^T\, V_b = {}^b_sR(t)^T\, [V]_\times^T\, V_b$$