Gershgorin's Theorem

$A \in \mathbb{C}^{n\times n}$, let $D_i = \{ z \in \mathbb{C} \mid |z - a_{ii}| \le \sum_{j\neq i} |a_{ij}| \}$
then all eigenvalue of A lie in $\bigcup_{i=1}^{n} D_i$

let (λ, x) be a (eigenvalue − eigenvector) pair of A
$k = \operatorname{argmax}_i |x_i|$

$\lambda x_k = \sum_j a_{kj} x_j$
$(\lambda - a_{kk}) x_k = \sum_{j\neq k} a_{kj} x_j$
$|(\lambda - a_{kk}) x_k| = |\sum_{j\neq k} a_{kj} x_j|$
$|\lambda - a_{kk}| = \dfrac{|\sum_{j\neq k} a_{kj} x_j|}{|x_k|} \le \dfrac{\sum_{j\neq k} |a_{kj}||x_j|}{|x_k|} \le \sum_{j\neq k} |a_{kj}|$

$|(a+bi)(c+di)|^2$
$= |ac-bd + (ad+bc)i|^2$
$= (ac-bd)^2 + (ad+bc)^2$
$= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2$

$|a+bi|\,|c+di|^2$
$= (a^2+b^2)(c^2+d^2)$
$= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2$

Diagonally-Dominant Matrix

$A \in \mathbb{C}^{n\times n}$. A is (strictly) diagonally-dominant if $|a_{ii}| \overset{(<)}{\le} \sum_{j\neq i} |a_{ij}| \ \forall i$

Theorem

A strictly diagonally-diagonal matix is nonsingular

$D_i = \{ z \in \mathbb{C} \mid |z - a_{ii}| \le \sum_{j\neq i} |a_{ij}| \}$
let λ be an eigenvalue of A. $\lambda \in \bigcup_{i=1}^{n} D_i$
WLOG, assume $\lambda \in D_k$
$-|\lambda| + |a_{kk}| \le |\lambda - a_{kk}| \le \sum_{j\neq i} |a_{kj}|$
$|\lambda| \ge |a_{kk}| - \sum_{j\neq i} |a_{kj}| > 0$
$\therefore$ A is nonsingular

Theorem

A symmetric diagonally-dominant real matrix with non-neg diagonals is PSD

A is symmetric real $\Rightarrow$ λ is real
$D_i = \{ z \in \mathbb{R} \mid |z - a_{ii}| \le \sum_{j\neq i} |a_{ij}| \}$
suppose $\lambda \in D_k$ $\quad |\lambda - a_{kk}| \le \sum_{j\neq k} |a_{kj}|$
if $\lambda < 0$ $\quad |\lambda - a_{kk}| = a_{kk} - \lambda \le \sum_{j\neq k} |a_{kj}|$

then $\lambda \geq a_{kk} - \sum_{j \neq k} |a_{kj}| \geq 0$

Contradict that $\lambda < 0$

Theorem

If A is strictly diagonally-dominant with positive diagonal elements

then the real part of its eigenvalues are positive

$|\lambda - a_{kk}| = |x + yi - a_{kk}| \leq \sum_{j \neq k} |a_{kj}|$

$|x - a_{kk}| \leq \sum_{j \neq k} |a_{kj}|$

if $x \leq 0$

$|x - a_{kk}| = a_{kk} - x \leq \sum_{j \neq k} |a_{kj}|$

$x \geq a_{kk} - \sum_{j \neq k} |a_{kj}| > 0$

Contradict that $x \leq 0$

$\therefore x > 0$