The Delta Method is a theorem that can be used to derive the distribution of a function of a asymptotically normal variable

let $\{X_n\}$ be a sequence of random variables s.t. $X_n \xrightarrow{d} \mathcal{N}(\mu, (\frac{\sigma}{\sqrt{n}})^2)$
and $g: \mathbb{R} \mapsto \mathbb{R}$ be a continuously differentiable

Then $g(X_n) \xrightarrow{d} \mathcal{N}(g(u), (\frac{\sigma g'(u)}{\sqrt{n}})^2)$

$$g(X_n) \approx g(u) + g'(u)(X_n - u)$$

$$\frac{g(X_n) - g(u)}{g'(u)} \approx \mathcal{N}(0, (\frac{\sigma}{\sqrt{n}})^2)$$

$$g(X_n) \approx \mathcal{N}(g(u), (\frac{\sigma g'(u)}{\sqrt{n}})^2)$$

Eg. Normalized Importance Weighting

$$\frac{\frac{1}{m}\sum_{i=1}^{m}\frac{\tilde{P}(X_i)}{Q(X_i)}f(X_i)}{\frac{1}{m}\sum_{i=1}^{m}\frac{\tilde{P}(X_i)}{Q(X_i)}} = \frac{\frac{1}{m}\sum_{i=1}^{m}w(X_i)f(X_i)}{\frac{1}{m}\sum_{i=1}^{m}w(X_i)} = \frac{A}{B}$$

$$\mu_A = E_{x\sim Q}[w(x)f(x)] \qquad \mu_B = E_{x\sim Q}[w(x)]$$

$$\sigma_A^2 = \frac{1}{m}\mathrm{Var}_{x\sim Q}(w(x)f(x)) \qquad \sigma_B^2 = \frac{1}{m}\mathrm{Var}_{x\sim Q}(w(x))$$

$$\sigma_{AB} = E_{x\sim Q}[AB] - E_{x\sim Q}[A]E_{x\sim Q}[B] = E_{x\sim Q}[w(x)^2 f(x)] - E_{x\sim Q}[w(x)f(x)]E_{x\sim Q}[w(x)]$$

$$\frac{A}{B} \approx \frac{\mu_A}{\mu_B} + \frac{1}{\mu_B}(A - \mu_A) - \frac{\mu_A}{\mu_B^2}(B - \mu_B)$$

$$\mathrm{Var}(\frac{A}{B}) \approx \frac{1}{\mu_B^2}\mathrm{Var}(A-\mu_A) + \frac{\mu_A^2}{\mu_B^4}\mathrm{Var}(B-\mu_B) - 2\frac{\mu_A}{\mu_B^3}\mathrm{Cov}(A-\mu_A, B-\mu_B)$$

$$= \frac{1}{\mu_B^2}\mathrm{Var}\left(\frac{1}{m}\sum_{i=1}^{m}w(x_i)f(x_i) - \mu_A\right) + \frac{\mu_A^2}{\mu_B^4}\mathrm{Var}\left(\frac{1}{m}\sum_{i=1}^{m}w(x_i) - \mu_B\right)$$

$$- 2\frac{1}{m}\frac{\mu_A}{\mu_B^3}\left\{E_{x\sim Q}[w(x)^2 f(x)] - E_{x\sim Q}[w(x)f(x)]E_{x\sim Q}[w(x)]\right\}$$

$$= \frac{1}{\mu_B^2}\frac{1}{m}E_{x\sim Q}[(w(x)f(x) - \mu_A)^2] + \frac{\mu_A^2}{\mu_B^4}\frac{1}{m}E_{x\sim Q}[(w(x) - \mu_B)^2] - 2\frac{\mu_A}{\mu_B^3}\frac{1}{m}\left\{E_{x\sim Q}[w(x)^2 f(x)] - \mu_A\mu_B\right\}$$

$$= \frac{1}{\mu_B^2}\frac{1}{m}\left\{E_{x\sim Q}[w(x)^2 f(x)^2] - \mu_A^2\right\}$$

$$+ \frac{\mu_A^2}{\mu_B^4}\frac{1}{m}\left\{E_{x\sim Q}[w(x)^2] - \mu_B^2\right\}$$

$$- 2\frac{\mu_A}{\mu_B^3}\frac{1}{m}\left\{E_{x\sim Q}[w(x)^2 f(x)] - \mu_A\mu_B\right\}$$

$$= \frac{1}{m}\frac{1}{\mu_B^2}\left\{ E_{x\sim Q}\left[w(x)^2 f(x)^2\right] + \frac{\mu_A^2}{\mu_B^2} E_{x\sim Q}\left[w(x)^2\right] - 2\frac{\mu_A}{\mu_B} E_{x\sim Q}\left[w(x)^2 f(x)\right]\right\}$$

$$= \frac{1}{m}\frac{1}{\mu_B^2} E_{x\sim Q}\left[\left(w(x) f(x) - \frac{\mu_A}{\mu_B} w(x)\right)^2\right]$$

$$= \frac{1}{m}\frac{1}{\mu_B^2} E_{x\sim Q}\left[w(x)^2\left(f(x) - \frac{\mu_A}{\mu_B}\right)^2\right]$$