properties

1° $(I+A)^{-1}$ and $(I-A)$ commute

$$
\begin{aligned}
&(I+A)^{-1}(I-A) \\
&= (I+A)^{-1}(-I-A+2I) \\
&= -I + 2(I+A)^{-1} \\
&= (-(I+A)+2I)(I+A)^{-1} \\
&= (I-A)(I+A)^{-1}
\end{aligned}
$$

theorem

1° Any orthogonal matrix $Q^TQ=I$ that does not have -1 eigenvalue can be expressed as

$$Q = (I+A)^{-1}(I-A)$$

for a skew-symmetric matrix $A=-A^T$

if the claim is true

$Q=(I+A)^{-1}(I-A)$ $\quad Q+AQ=I-A$ $\quad Q-I=-AQ-A$ $\quad A=(I-Q)(I+Q)^{-1}=(I+Q)^{-1}(I-Q)$

proof: let $A=(I+Q)^{-1}(I-Q)$

1° $A+QA=I-Q$

$QA+Q=I-A$

$Q=(I-A)(I+A)^{-1}=(I+A)^{-1}(I-A)$

2°
$$
\begin{aligned}
A^T &= (I-Q)^T(I+Q)^{-T} \\
&= (I-Q^{-1})(I+Q^{-1})^{-1} \\
&= (I-Q^{-1})QQ^{-1}(I+Q^{-1})^{-1} \\
&= (Q-I)[(I+Q^{-1})Q]^{-1} \\
&= (Q-I)(Q+I)^{-1} = -(I-Q)(I+Q)^{-1} \\
&= -(I+Q)^{-1}(I-Q) = -A
\end{aligned}
$$
A is skew-symmetric

2° Any skew-symmetric matrix $A^T=-A$ can be expressed as

$$A=(I+Q)^{-1}(I-Q)$$

where $Q^TQ=I$, Q does not have -1 eigenvalue

if the claim is true

$A=(I+Q)^{-1}(I-Q)$ $\quad A+QA=I-Q$ $\quad Q=(I-A)(I+A)^{-1}=(I+A)^{-1}(I-A)$

proof let $Q=(I+A)^{-1}(I-A)$

1° $Q+AQ=I-A$

$A=(I-Q)(I+Q)^{-1}=(I+Q)^{-1}(I-Q)$

2°
$$
\begin{aligned}
Q^T &= (I-A)^T(I+A)^{-T} \\
&= (I+A)(I-A)^{-1} \\
&= (I-A)^{-1}(I+A) \\
&= Q^{-1}
\end{aligned}
$$

3° If $Qv=-v$

$(I+A)^{-1}(I-A)v=-v$

$v-Av=-v-Av$

$v=-v$

$\therefore Q$ does not have -1 e.v

Cayley Transform

Cayley Transform defines a one-to-one correspondence

between
- skew-symmetric matrices
- orthonormal matrices that does not have -1 eigenvalue

relationship defined as

$$\begin{cases} Q = (I+A)^{-1}(I-A) \\ A = (I+Q)^{-1}(I-Q) \end{cases}$$

properties

$(iI+A)^{-1}$ and $(iI-A)$ commutes

$$(iI+A)^{-1}(iI-A)$$
$$= (iI+A)^{-1}(-iI-A+2iI)$$
$$= -I + 2i(iI+A)^{-1}$$
$$= [-(iI+A)+2iI](iI+A)^{-1}$$
$$= (iI-A)(iI+A)^{-1}$$

$\frac{1}{i} = i$

$$\frac{1}{i}(iI+A)^{-1} i(iI-A)$$
$$= (-i\,iI+iA)^{-1}(-I-Ai)$$
$$= (A+I)^{-1}(-I-Ai)$$

Theorem

1° Any unitary matrix $Q^HQ=I$ that does not have 1 eigenvalue can be expressed as

$$Q = -(iI+A)^{-1}(iI-A) =$$

where A is skew-hermetian $A=-A^H$ (real part of A is skew-symmetric, img part of A is symmetric)

if the claim is true

$Q=-(iI+A)^{-1}(iI-A)$ $\quad iQ+AQ=-iI+A$ $\quad i(Q+I)=A(I-Q)$ $\quad A=i(I+Q)(I-Q)^{-1}$

proof: let $A=i(I+Q)(I-Q)^{-1}$

1° $A-AQ=iI+iQ$

$-(A+iI)Q=iI-A$

$Q=-(iI+A)^{-1}(iI-A)$

2° $A^H=i(I-Q^H)^{-1}(I+Q^H)$

$= i(I-Q^{-1})^{-1}Q^{-1}Q(I+Q^{-1})$

$= i(Q-I)^{-1}(Q+I)$

$= -A$

2° Any Skew-Hermitian matrix $A^H=-A$ can be expessed as

$$A = i(I+Q)(I-Q)^{-1}$$

where $Q^HQ=I$, Q does not have 1 eigenvalue

if the claim is true

$A-AQ=iI+iQ$ $\quad (-A-iI)Q=iI-A$ $\quad Q=-(iI+A)^{-1}(iI-A)$

proof: let $Q=-(iI+A)^{-1}(iI-A)$

1° $-iQ-AQ=iI-A$

$A-AQ=iI+iQ$

$A=i(I+Q)(I-Q)^{-1}$

2° $Q^H=-(iI^H-A^H)(iI^H+A^H)^{-1}$

$=-(-iI+A)(-iI-A)^{-1}$

$=-(iI-A)(iI+A)^{-1}$

$= Q$

Cayley Transform

Cayley Transform defines a one-to-one correspondence

Skew-Hermitian

between unitary matrix that does not have 1 eigenvalue

relationship defined as

$$\begin{cases} Q = -(iI + A)^{-1}(iI - A) \\ A = i(I+Q)(I-Q)^{-1} \end{cases}$$