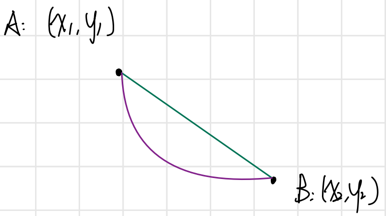
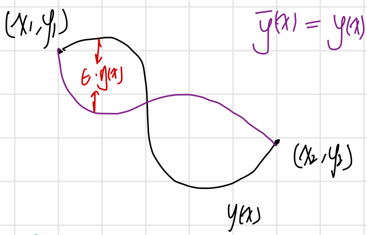


minimize the distance:

$$\min_y \; I(y) = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$$

minimize the time from A to B under gravity (Brachistochrone problem)

$$\min_y \; I(y) = \int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{v}\, dx = \int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2g(y_1-y)}}\, dx$$



$$\bar{y}(x) = y(x) + \epsilon \cdot \eta(x)$$

$$\min_y \int_{x_1}^{x_2} F(x, y(x), y'(x))\, dx$$

s.t. $y(x_1) = y_1$, $y(x_2) = y_2$

let
- $y(x)$ be the optimal solution
- $\bar{y}(x) = y(x) + \epsilon \cdot \eta(x)$
- η be an arbitrary variation satisfying $\eta(x_1) = 0$, $\eta(x_2) = 0$

$$I[\bar{y}] = \int_{x_1}^{x_2} F(x, \bar{y}(x), \bar{y}'(x))\, dx$$

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0 \qquad \forall\, y$$

$$= \left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} \int_{x_1}^{x_2} F(x, \bar{y}(x), \bar{y}'(x))\, dx$$

$$= \int_{x_1}^{x_2} \left.\left(\frac{\partial F}{\partial x} \cdot \underbrace{\frac{dx}{d\epsilon}}_{0} + \frac{\partial F}{\partial \bar{y}} \cdot \frac{d\bar{y}}{d\epsilon} + \frac{\partial F}{\partial \bar{y}'} \cdot \frac{d\bar{y}'}{d\epsilon} \right)\right|_{\epsilon=0} dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \cdot y + \frac{\partial F}{\partial y'} \cdot y'\, dx$$

1st variation weak form since we have y' in the integral

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \cdot y + \underbrace{\left.\frac{\partial F}{\partial y'} \cdot y\right|_{x=x_1}^{x=x_2}}_{y(x_1) = y(x_2) = 0} - \int_{x_1}^{x_2} y \cdot \frac{d}{dx} \frac{\partial F}{\partial y'}\, dx$$

$$= \int_{x_1}^{x_2} y \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) dx$$

$$= 0$$

Since y is arbitrary, must have $\underbrace{\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0}$ $\forall x$

↓ Euler-Lagrange Equation

Eg $\min_y I[y] = \int_0^1 (y' - y)^2\, dx$

s.t. $y(0) = 0 \quad y(1) = 2$

$F(x, y, y') = (y' - y)^2$

Euler-Lagrange $\frac{\partial F}{\partial y} - \frac{d}{dx} \cdot \frac{\partial F}{\partial y'} = 0$

$$2(y - y') - \frac{d}{dx} 2(y' - y) = 0$$

$y'' - y = 0 \Rightarrow y = e^{rx} \Rightarrow r^2 - r = 0 \quad r = \pm 1$

$\Rightarrow y = c_1 e^x + c_2 \cdot e^{-x}$

$$\left.\begin{cases} y(0) = C_1 + C_2 = 0 \\ y(1) = C_1 \cdot e + C_2 \cdot \frac{1}{e} = 2 \end{cases}\right\} \Rightarrow C_1 \cdot \left(e - \frac{1}{e}\right) = 2$$

$$y(x) = 2 \frac{e^x - e^{-x}}{\left(e - \frac{1}{e}\right)} = 2 \frac{\sinh(x)}{\sinh(1)} \qquad \sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$$