Bellman - Gronwall Lemma

suppose $\begin{cases} u(\cdot)\ \ k(\cdot) \text{ are } \begin{cases} \text{real-valued} \\ \text{piecewise continuous} \\ \geq 0 \text{ on } \mathbb{R}_+ \end{cases} \\ c_1 \geq 0 \quad t_0 \geq 0 \end{cases}$

if $u(t) \overset{(\geq)}{\leq} c_1 + \int_{t_0}^{t} k(\tau) u(\tau)\, d\tau$

then $u(t) \overset{(\geq)}{\leq} c_1 \cdot \exp\left(\int_{t_0}^{t} k(\tau)\, d\tau\right)$

let $B(t) = c_1 + \int_{t_0}^{t} k(\tau) u(\tau)\, d\tau$

we have $u(t) \leq B(t)$

$$u(t)\, k(t) \exp\left(-\int_{t_0}^{t} k(\tau)\, d\tau\right) \leq B(t)\, k(t) \exp\left(-\int_{t_0}^{t} k(\tau)\, d\tau\right)$$

$$u(t)\, k(t) \exp\left(-\int_{t_0}^{t} k(\tau)\, d\tau\right) - B(t)\, k(t) \exp\left(-\int_{t_0}^{t} k(\tau)\, d\tau\right) \leq 0$$

$$\frac{d}{dt}\left[B(t) \exp\left(-\int_{t_0}^{t} k(\tau)\, d\tau\right)\right] \leq 0$$

$$B(t) \exp\left(-\int_{t_0}^{t} k(\tau)\, d\tau\right) \leq B(t_0) \exp\left(-\int_{t_0}^{t_0} k(\tau)\, d\tau\right) = c_1$$

$$u(t) \leq B(t) \leq c_1 \cdot \exp\left(\int_{t_0}^{t} k(\tau)\, d\tau\right)$$

Application: Uniqueness of Solution to ODE

$x(t)$ is solution to $\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \end{cases}$ f is Lipschitz in x $\quad \|f(x,t) - f(y,t)\|_2 \leq L(t)\, \|x-y\|_2$

Then the solution is unique

assume there're two solutions $x(t)$ and $y(t)$

$$\begin{cases} \dot{x} = f(x,t) & x(t_0) = x_0 \\ \dot{y} = f(y,t) & y(t_0) = x_0 \end{cases} \Rightarrow \begin{array}{l} x(t) = x_0 + \int_0^t f(x(\tau),\tau)\, d\tau \\ y(t) = x_0 + \int_0^t f(y(\tau),\tau)\, d\tau \end{array}$$

$$\begin{aligned} \|x(t) - y(t)\| &= \left\| \int_0^t f(x(\tau),\tau) - f(y(\tau),\tau)\, d\tau \right\| \\ &\leq \int_0^t \|f(x(\tau),\tau) - f(y(\tau),\tau)\|\, d\tau \\ &\leq \hat{L} \int_0^t \|x(\tau) - y(\tau)\|\, d\tau \end{aligned}$$

apply Bellman - Gronwall with $C = 0$ $\mu(t) = \hat{L}$

$$\|x(t) - y(t)\| \leq 0 \cdot \exp\left[\int_{t_0}^{t} \hat{L} \, d\tau \right] = 0$$