- Constrained piece-wise linear minimization

$$\min_{x,z} \max\left(A\begin{bmatrix}x\\z\end{bmatrix}+b\right) = f(x,z)$$
$$\text{s.t. } z=y$$

$$f(x^*,z^*) = g(v^*) = \inf_{x,z} f(x,z) + v^{*T}(z-y)$$
$$\le f(x^*,z^*) + v^*(z^*-y)$$
$$= f(x^*,z^*)$$

$$\frac{\partial f(x^*,z^*)}{\partial y} = \frac{\partial g(v^*)}{\partial y} = -v^*$$

the dual variable gives the local sensitivity to variable constraint

$$\min_{x,t,z} \; t$$
$$\text{s.t. } A\begin{bmatrix}x\\z\end{bmatrix}+b \le t\cdot 1$$
$$z=y$$

$$\Downarrow$$

$$\min_{x,z,t} c\cdot t - \sum \log\left(t\cdot 1 - A\begin{bmatrix}x\\z\end{bmatrix} - b\right)$$
$$\text{s.t. } z=y$$

$$\Downarrow$$

$$\min_{x,z,t} [0,0,c]\begin{bmatrix}x\\z\\t\end{bmatrix} - \sum \log\left([-A\ 1]\begin{bmatrix}x\\z\\t\end{bmatrix} - b\right)$$
$$\text{s.t. } [0,1,0]\begin{bmatrix}x\\z\\t\end{bmatrix} = y$$

$$\mathcal{L}(x,z,t,v) = [0,0,c]\begin{bmatrix}x\\z\\t\end{bmatrix} - \sum \log\left([-A\ 1]\begin{bmatrix}x\\z\\t\end{bmatrix} - b\right) + v^T\left([0,1,0]\begin{bmatrix}x\\z\\t\end{bmatrix} - y\right)$$

$$\nabla_{xzt} f = [0,0,c] - [-A\ 1]^T \, 1/\left([-A\ 1]\begin{bmatrix}x\\z\\t\end{bmatrix} - b\right)$$

$$\nabla^2_{xzt} f = [-A\ 1]^T \operatorname{diag}\left([-A\ 1]\begin{bmatrix}x\\z\\t\end{bmatrix} - b\right)^{-2} [-A\ 1]$$

$$\begin{cases} \nabla\mathcal{L}(x+dx, z+dz, t+dt) \approx \nabla f(x,z,t) + \nabla^2 f(x,z,t)\begin{bmatrix}dx\\dz\\dt\end{bmatrix} + v\begin{bmatrix}0\\1\\0\end{bmatrix} := 0 & \text{primal residual} \\ [0\ 1\ 0]\begin{bmatrix}dx\\dz\\dt\end{bmatrix} = 0 & \text{primal residual} \end{cases}$$

$$\begin{bmatrix} [H] & \begin{bmatrix}0\\1\\0\end{bmatrix} \\ [0\ 1\ 0] & [0] \end{bmatrix} \begin{bmatrix} \begin{bmatrix}dx\\dz\\dt\end{bmatrix} \\ [v] \end{bmatrix} = \begin{bmatrix} -g \\ 0 \end{bmatrix}$$

$$\text{row 2} - [0\ 1\ 0] H^{-1} \text{row 1}: \quad -[0\ 1\ 0]H^{-1}\begin{bmatrix}0\\1\\0\end{bmatrix} v = [0\ 1\ 0]H^{-1} g \quad \Big\} \text{ solve for dual variable}$$

$$H\begin{bmatrix}dx\\dz\\dt\end{bmatrix} = -g - \begin{bmatrix}0\\1\\0\end{bmatrix} v \quad \Big\} \text{ solve for } \begin{bmatrix}dx\\dz\\dt\end{bmatrix}$$

$$\min \quad t f_0(x) - \sum_i \log(-f_i(x))$$
$$\text{s.t.} \quad Ax = b$$

$$L(x,v) = t \cdot f_0(x) - \sum_i \log(-f_i(x)) + v^T(Ax-b)$$

$$\begin{cases} \nabla L_x(x,v) = t\,\nabla f_0(x) + \sum_i \frac{1}{-f_i(x)} \cdot \nabla f_i(x) + A^T v \\ Ax - b = 0 \end{cases}$$

$$\nabla f_0(x^*(t)) + \sum_i \underbrace{\frac{1}{-t\, f_i(x^*(t))}}_{\lambda_i} \nabla f_i(x^*(t)) + A^T \underbrace{(v/t)}_{v} = 0$$