· Coupled piece-wise linear minimization

$$\min_{x_1\ x_2} \ \max\left(A_1\begin{bmatrix}x_1\\y_1\end{bmatrix}+b_1\right)+\max\left(A_2\begin{bmatrix}x_2\\y_2\end{bmatrix}+b_2\right)$$

· Primal decomposition

fix y and define:

subproblem 1. $\min_{x_1} \ \max\left(A_1\begin{bmatrix}x_1\\y\end{bmatrix}+b_1\right)$

subproblem 2. $\min_{x_2} \ \max\left(A_2\begin{bmatrix}x_2\\y\end{bmatrix}+b_2\right)$

and optimal values for subproblems $\phi_1(y)$ $\phi_2(y)$

master problem $\min_y \ \phi_1(y)+\phi_2(y)$

$\phi_1(y)+\phi_2(y)=\min_{x_1\ x_2} \ \max\left(A_1\begin{bmatrix}x_1\\y\end{bmatrix}+b_1\right)+\max\left(A_2\begin{bmatrix}x_2\\y\end{bmatrix}+b\right)$ is a convex function

$$\min_{x_1,z} \ A_1\begin{bmatrix}x_1\\z\end{bmatrix}+b_1 \qquad -v^*=\frac{\partial g(v^*)}{\partial y}$$

s.t. $z=y$ (dual variable)

solve master problem by bisection

if $v_1^*+v_2^* > 0$; $\partial(\phi_1^*+\phi_2^*)/\partial y<0$; increase y

else decrease y

· Dual decomposition

$$\min \ \max\left(A_1\begin{bmatrix}x_1\\y\end{bmatrix}+b_1\right)+\max\left(A_2\begin{bmatrix}x_2\\y\end{bmatrix}+b_2\right)$$

$$\Downarrow$$

$$\min \ \max\left(A_1\begin{bmatrix}x_1\\y_1\end{bmatrix}+b_1\right)+\max\left(A_2\begin{bmatrix}x_2\\y_2\end{bmatrix}+b_2\right)$$

s.t. $y_1=y_2$

$$L(x_1,x_2,y_1,y_2,v)=\max\left(A_1\begin{bmatrix}x_1\\y_1\end{bmatrix}+b_1\right)+\max\left(A_2\begin{bmatrix}x_2\\y_2\end{bmatrix}+b_2\right)+v^T(y_1-y_2)$$

dual $g(v)=\inf_{\substack{x_1,x_2\\y_1,y_2}} L(x_1,x_2,y_1,y_2,v)$

$$=\inf_{x_1,y_1} \ \max\left(A_1\begin{bmatrix}x_1\\y_1\end{bmatrix}+b_1\right)+v^Ty_1 \ + \ \inf_{x_2,y_2} \ \max\left(A_2\begin{bmatrix}x_2\\y_2\end{bmatrix}+b_2\right)-v^Ty_2$$

subproblem 1 $g_1(v)=\inf_{x_1\ y_1} \ \max\left(A_1\begin{bmatrix}x_1\\y_1\end{bmatrix}+b_1\right)+v^Ty_1$

subproblem 2 $g_2(v)=\inf_{x_2\ y_2} \ \max\left(A_2\begin{bmatrix}x_2\\y_2\end{bmatrix}+b_2\right)-v^Ty_1$

master (dual) problem max. $g_1(v) + g_2(v)$

$\Downarrow$

min $-g_1(v) - g_2(v)$

a gradient for (negative) dual problem is $y_2 - y_1$

Solve the master problem with bisection

if $\quad d\,{-g}/dv = y_2 - y_1 > 0$: decrease v

else $\quad$ increase v