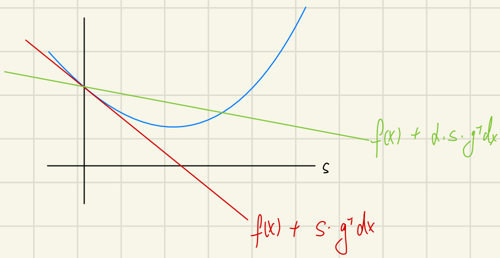
- non-negative QP

$$\min_x \ \tfrac{1}{2}x^TPx + q^Tx + r$$
s.t. $x \geq 0$

$\Downarrow$ log barrier

$$\min_x \ t\left(\tfrac{1}{2}x^TPx + q^Tx + r\right) - \sum \log(x)$$

$$\nabla f(x) = t(Px+q) - 1/x$$
$$\nabla^2 f(x) = tP + \mathrm{diag}(1/x^2)$$
$$\nabla^2 f(x)\, dx = -\nabla f(x)$$



$$\min_x \ t f_0(x) - \sum_i \log(-f_i(x))$$
s.t. $Ax = b$

$$\mathcal{L}(x,v) = t f_0(x) - \sum_i \log(-f_i(x)) + v^T(Ax-b)$$
$$\nabla_x \mathcal{L} = t\nabla f_0(x) + \sum_i \frac{1}{-f_i(x)}\nabla f_i(x) + A^Tv = 0$$
$$\nabla f_0(x) + \sum_i \underbrace{\frac{1}{-t f_i(x)}}_{\tilde{\lambda}_i} \nabla f_i(x) + A^T\underbrace{(v/t)}_{\tilde{v}} = 0$$

$\tilde{x}$ minimizes the Lagrangian
$$\mathcal{L}(x,\tilde{\lambda},\tilde{v}) = f_0(x) + \sum_i \lambda_i f_i(x) + \tilde{v}^T(Ax-b)$$
$$g(\tilde{\lambda},\tilde{v}) = \mathcal{L}(\tilde{x},\tilde{\lambda},\tilde{v})$$

$$p^* \geq g(\tilde{\lambda},\tilde{v}) = \mathcal{L}(\tilde{x},\tilde{\lambda},\tilde{v}) = f_0(\tilde{x}) - m/t$$

$$p^* = f(x^*) \geq f(x^*) + \sum_i \lambda_i f_i(x^*) + v^T(Ax^*-b) \geq \inf_x f(x) + \sum_i \lambda_i f_i(x) + v^T(Ax-b)$$

Non-negative matrix factorization QP

$$\min_{X,Y} \ \|A - XY\|_F^2 \qquad A \in \mathbb{R}^{m\times n} \quad X \in \mathbb{R}^{m\times k} \quad Y \in \mathbb{R}^{k\times n}$$
s.t. $X_{ij} \geq 0$, $Y_{ij} \geq 0$

$$\|A - XY\|_F^2 = \mathrm{tr}\left[(A-XY)^T(A-XY)\right]$$
$$= \mathrm{tr}\left[A^TA - A^TXY - (XY)^TA + (XY)^TXY\right]$$
$$= \mathrm{tr}\left[A^TA - 2A^TXY + Y^TX^TXY\right]$$

$$= \mathrm{tr}[\underbrace{A^TA}_{n\times n}] - 2\,\mathrm{tr}[\underbrace{A^TX}_{n\times k}Y] + \mathrm{tr}[\underbrace{Y^T}_{n\times k}\underbrace{X^TX}_{k\times k}\underbrace{Y}_{k\times n}]$$
$$= \sum_j^n \|A[:,j]\|_2^2 - 2(A^TX)[j,:]^T Y[:,j] + Y[:,j]^TX^TXY[:,j]$$

$$= \mathrm{tr}[\underbrace{AA^T}_{m\times m}] - 2\,\mathrm{tr}[\underbrace{X}_{m\times k}\underbrace{YA^T}_{k\times m}] + \mathrm{tr}[\underbrace{X}_{m\times k}\underbrace{YY^T}_{k\times k}\underbrace{X^T}_{k\times m}]$$
$$= \sum_j^m \|A[j,:]\|_2^2 - 2X[j,:]^T(YA^T)[:,j] + X[j,:]^TYY^TX[j,:]$$

for $j = 1:n$
$$Y[:,j] = \arg\min_y \ y^TX^TXy - 2(A^TX)[j,:]^Ty + \|A[:,j]\|_2^2$$
s.t. $y \geq 0$

for $j = 1:m$
$$X[j,:] = \arg\min_x \ x^TYY^Tx - 2(YA^T)[:,j]^Tx + \|A[j,:]\|_2^2$$
s.t. $x \geq 0$