· Trust region constrained convex quadratic programming.

$$\min.\ \tfrac{1}{2}x^TPx + q^Tx + r$$

$$\text{s.t.}\quad \|x\|_\infty \le 1 \Rightarrow -1 \le x \le 1 \Rightarrow \begin{cases} -x-1 \le 0 \\ x-1 \le 0 \end{cases}$$

$$\|x - x^{(k)}\| \le \rho \Rightarrow -\rho \le x - x^{(k)} \le \rho \Rightarrow \begin{cases} -x + x^{(k)} - \rho \le 0 \\ x - x^{k} - \rho \le 0 \end{cases}$$

$$-\rho + x^{(k)} \le x \le \rho + x^{(k)}$$

$$\Downarrow$$

$$\min.\ \tfrac{1}{2}x^TPx + q^Tx + r$$

$$\begin{aligned} \text{s.t.}\quad & -x-1 \le 0 \\ & x-1 \le 0 \\ & -x+a \le 0 \qquad a = x^{(k)} - \rho \\ & x-b \le 0 \qquad b = x^{(k)} + \rho \end{aligned}$$

$$\Downarrow \text{ log barrier}$$

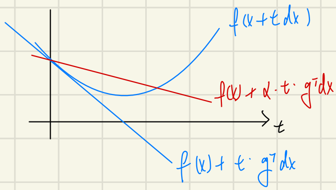
$$\min.\ t\left(\tfrac{1}{2}x^TPx + q^Tx + r\right) - \log(x+1) - \log(1-x) - \log(x-a) - \log(b-x)$$

$$\nabla f(x) = t(Px+q) - \frac{1}{x+1} + \frac{1}{1-x} - \frac{1}{x-a} + \frac{1}{b-x}$$

$$\nabla^2 f(x) = tP + \mathrm{diag}\left(\frac{1}{(x+1)^2}\right) + \mathrm{diag}\left(\frac{1}{(1-x)^2}\right) + \mathrm{diag}\left(\frac{1}{(x-a)^2}\right) + \mathrm{diag}\left(\frac{1}{(b-x)^2}\right)$$

$$\nabla f(x+\Delta x) \approx \nabla f(x) + \nabla^2 f(x)\Delta x := 0$$

$$\Delta x = -\nabla^2 f(x)^{-1}\nabla f(x)$$



$$\min.\ t \cdot f_0(x) - \sum_i \log -f_i(x)$$

$$\text{s.t.}\quad Ax = b$$

$$\tilde{\mathcal{L}}(x,v) = tf_0(x) - \sum_i \log -f_i(x) + v^T(Ax-b)$$

$$\nabla_x \tilde{\mathcal{L}} = t\nabla f_0(\tilde{x}) + \sum_i \frac{1}{-f_i(\tilde{x})}\nabla f_i(\tilde{x}) + A^Tv = 0$$

$$\nabla f_0(\tilde{x}) + \sum_i \underbrace{\frac{1}{-tf_i(\tilde{x})}}_{\tilde{\lambda}_i}\nabla f_i(\tilde{x}) + A^T\underbrace{v/t}_{\tilde{v}} = 0$$

$$\begin{aligned} p^* &= f_0(x^*) \\ &\ge f_0(x^*) + \sum_i \lambda_i f_i(x^*) + v^T(Ax^*-b) \\ &\ge \inf_x f_0(x) + \sum_i \lambda_i f_i(x) + v^T(Ax^*-b) \\ &= g(\lambda, v) \end{aligned}$$

$\tilde{x}$ minimize the lagrangian $f_0(x) + \sum_i \tilde{\lambda}_i f_i(x) + \tilde{v}(Ax-b)$

$$\begin{aligned} p^* &\ge g(\tilde{\lambda}, \tilde{v}) \\ &= \mathcal{L}(\tilde{x}, \tilde{\lambda}, \tilde{v}) \\ &= f_0(\tilde{x}) + \sum_i \frac{1}{-tf_i(\tilde{x})} f_i(\tilde{x}) + \tilde{v}(A\tilde{x}-b) = f_0(\tilde{x}) - m/t \end{aligned}$$

- Non convex quadratic programming

$$\min.\ f(x) = \frac{1}{2} x^T P x + q^T x + r \qquad (P \not\succeq 0)$$
$$\text{s.t.}\quad \|x\|_\infty \le 1$$

$$\nabla f(x) = Px + q, \qquad \nabla^2 f(x) = P$$

Convex approximation

$$\hat f_{x^{(k)}}(x) = f(x^{(k)}) + (Px^{(k)}+q)^T (x - x^{(k)}) + \frac{1}{2}(x - x^{(k)})^T P_+ (x - x^{(k)})$$
$$= f(x^{(k)}) + (Px^{(k)}+q)^T x - (Px^{(k)}+q)^T x^{(k)} + \frac{1}{2} x^T P_+ x - x^T P_+ x^{(k)} + \frac{1}{2} x^{(k)T} P_+ x^{(k)}$$
$$= \frac{1}{2} x^T P_+ x + (Px^{(k)} + q - P_+ x^{(k)})^T x + \frac{1}{2} x^{(k)T} P_+ x^{(k)} - (Px^{(k)}+q)^T x^{(k)} + f(x^{(k)})$$

Solve the trust region constrained quadratic programming

$$\min.\ \hat f(x) = \frac{1}{2} x^T \tilde P x + \tilde q^T x + \tilde r \qquad \tilde P = P_+$$
$$\text{s.t.}\quad \|x - x^{(k)}\|_\infty \le \rho \qquad \tilde q = Px^{(k)} + q - P_+ x^{(k)}$$
$$\|x\|_\infty \le 1 \qquad \tilde r = \frac{1}{2} x^{(k)T} P_+ x^{(k)} - (Px^{(k)}+q)^T x^{(k)} + f(x^{(k)})$$

Trust region update

predicted decrease: $\hat\delta = \hat f(x^{(k)}) - \hat f(\hat x)$

actual decrease: $\delta = f(x^{(k)}) - f(\hat x)$