$$\begin{aligned} \min.\ & f_0(x) \\ s.t.\ & f_i(x) \le 0 \quad i=1,\dots,m \\ & Ax=b \end{aligned}$$

Log Barrier

$$\begin{aligned} \min.\ & t f_0(x) - \sum_i \log -f_i(x) \quad \Rightarrow \tilde{x} \\ s.t.\ & Ax=b \end{aligned}$$

$$p^* = f_0(x^*) = \inf_{\substack{f_i(x)\le 0 \\ Ax=b}} f_0(x)$$

$$\ge f_0(x^*) + \sum_i \lambda_i f_i(x^*) + v^T(Ax^*-b) \qquad \lambda_i \ge 0$$

$$\ge \inf_x f_0(x) + \sum_i \lambda_i f_i(x) + v^T(Ax-b) \qquad \lambda_i \ge 0$$

$$= g(\lambda, v)$$

$$\tilde{L}(x,v) = t f_0(x) - \sum_i \log -f_i(x) + v^T(Ax-b)$$

$$\nabla_x \tilde{L} = t\nabla f_0(x) + \sum_i \frac{1}{-f_i(x)} \nabla f_i(x) + A^T v := 0$$

$$\nabla f_0(x) + \sum_i \underbrace{\frac{1}{-t f_i(x)}}_{\tilde{\lambda}_i} \nabla f_i(x) + A^T \underbrace{(v/t)}_{\tilde{v}} = 0$$

$\tilde{x}$ minimize the Lagrangian $f(x) + \sum_i \tilde{\lambda}_i f_i(x) + \tilde{v}^T(Ax-b)$

$$\tilde{\lambda}_i = \frac{1}{-t f_i(\tilde{x})} \qquad \tilde{v} = v/t$$

$\therefore\ g(\tilde{\lambda}, \tilde{v}) = f(\tilde{x}) + \sum_i \tilde{\lambda}_i f_i(\tilde{x}) + \tilde{v}^T(A\tilde{x}-b)$ is a lower bound

$$\xrightarrow{\text{modified KKT condition}} \quad \begin{cases} \nabla f(\tilde{x}) + \sum_i \tilde{\lambda}_i \nabla f_i(\tilde{x}) + A^T \tilde{v} = 0 \\ -\tilde{\lambda}_i f_i(\tilde{x}) = 1/t \\ A\tilde{x} = b \end{cases}$$

$$p^* \ge g(\tilde{\lambda}, \tilde{v}) = f(\tilde{x}) + \sum_i \frac{1}{-t f_i(\tilde{x})} \cdot f_i(\tilde{x}) + \tilde{v}^T(A\tilde{x}-b)$$

$$= f(\tilde{x}) - m/t \quad \text{duality gap}$$

· Primal – Dual Search direction

$$r_t(x,\lambda,v) = \begin{bmatrix} \nabla f_0(x) + Df(x)^T\lambda + A^T v \\ -\operatorname{diag}(\lambda) f(x) - \frac{1}{t}\mathbb{1} \\ Ax-b \end{bmatrix} \begin{matrix} \} \text{ dual residual} \\ \} \text{ centrality residual} \\ \} \text{ primal residual} \end{matrix}$$

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} \qquad Df(x) = \begin{bmatrix} - \nabla f_1(x) - \\ \vdots \\ - \nabla f_m(x) - \end{bmatrix}$$

if x, λ, v satisfies $r_t(x,\lambda,v) = 0$ (and $f_i(x) < 0$),

then $x = x^*(t)$ $\lambda = \lambda^*(t)$ $v = v^*(t)$ solves the log-barrier

x is primal feasible, λ, v are dual feasible with duality gap m/t

$$y = (x, \lambda, v)$$

$$r_t(y+\Delta y) \approx r_t(y) + Dr_t(y)\Delta y := 0$$

$$\begin{bmatrix} \nabla^2 f_0(x) + \sum_i \lambda_i \nabla^2 f_i(x) & Df(x)^T & A^T \\ -\operatorname{diag}(\lambda) Df(x) & -\operatorname{diag}(f(x)) & 0 \\ A & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta v \end{bmatrix} = - \begin{bmatrix} \nabla f_0(x) + Df(x)^T\lambda + A^T v \\ -\operatorname{diag}(\lambda) f(x) - \frac{1}{t}\mathbb{1} \\ Ax-b \end{bmatrix} \begin{matrix} n \\ m \\ m \end{matrix}$$

Solve for search direction

- Surrogate duality gap

for any x, λ s.t. $f_i(x) \le 0$ $\lambda_i \ge 0$

$\hat{\eta}(x,\lambda) = -f(x)^T\lambda$

when x is primal feasible ($f_i(x) \le 0$, $Ax = b$) λ is dual feasible ($\nabla f_0(x) + Df(x)^T\lambda + A^T v = 0$)

then $\hat{\eta}(x,\lambda)$ is the duality gap

$\hat{\eta}(x,\lambda) = m/t$ t corresponds to current duality gap is $m/\hat{\eta}$

- Primal-dual interior point method

given x s.t. $f_i(x) < 0$ $\lambda > 0$ $u > 1$ $\epsilon_{feas} > 0$ $\epsilon > 0$

repeat

determine t: $t = u\, m/\hat{\eta}$

compute primal-dual search direction

line search and update

until $\|r_{pri}\|_2 \le \epsilon_{fea}$ $\|r_{dual}\|_2 \le \epsilon_{fea}$ $\hat{\eta} < \epsilon$

- line search

$s = 1$

1° $\lambda + s\Delta\lambda \ge 0$

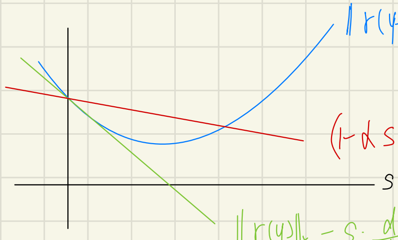
2° $f_i(x + s\Delta x) < 0$

$$\frac{d}{ds}\|r(y+s\Delta y)\|_2^2\Big|_{s=0} = 2r(y)^T Dr(y)\Delta y$$

$$= 2r(y)^T(-r(y)) = -2\|r(y)\|_2^2$$

$$\frac{d}{dt}\|r(y+s\Delta y)\|_2^2\Big|_{s=0} = 2\|r(y)\|_2 \cdot \frac{d}{dt}\|r(y+s\Delta y)\|_2\Big|_{s=0}$$

$$\frac{d}{dt}\|r(y+s\Delta y)\|_2\Big|_{s=0} = -\|r(y)\|_2$$



eg. Linear Programming

$$\min.\ c^T x$$
$$\text{s.t.}\ x \geq 0$$
$$Ax = b \qquad \mathcal{L}(x,\lambda,\nu) = c^T x - x^T\lambda + \nu^T(Ax-b) = 0$$

modified KKT condition

$$\begin{cases} \nabla_x \mathcal{L} = c - \lambda + A^T\nu = 0 \\ \lambda_i x_i = 1/t \\ Ax = b \end{cases}$$

primal dual search direction

$$r(y) = r(x,\lambda,\nu) = \begin{bmatrix} c-\lambda+A^T\nu \\ \mathrm{diag}(\lambda)x - 1/t \cdot \mathbf{1} \\ Ax-b \end{bmatrix}$$

$$r(y+\Delta y) \approx r(y) + Dr(y)\cdot\Delta y := 0$$

$$\begin{bmatrix} 0 & -I & A^T \\ \mathrm{diag}(\lambda) & \mathrm{diag}(x) & 0 \\ A & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta\nu \end{bmatrix} = -\begin{bmatrix} c-\lambda+A^T\nu \\ \mathrm{diag}(\lambda)x - 1/t\cdot\mathbf{1} \\ Ax-b \end{bmatrix} = \begin{bmatrix} -r_d \\ -r_c \\ -r_p \end{bmatrix}$$

$$\text{row3} - A\,\mathrm{diag}(\lambda)^{-1}\ \text{row2}: \begin{bmatrix} 0 & -I & A^T \\ \mathrm{diag}(\lambda) & \mathrm{diag}(x) & 0 \\ 0 & -A\,\mathrm{diag}(1/\lambda)\,\mathrm{diag}(x) & 0 \end{bmatrix}$$

$$\text{row3} - A\,\mathrm{diag}(\lambda)^{-1}\ \text{row2} - A\,\mathrm{diag}(\lambda)^{-1}\mathrm{diag}(x)\ \text{row1}:$$

$$-A\,\mathrm{diag}(\lambda)^{-1}\,\mathrm{diag}(x)\,A^T\,\Delta\nu = -r_p + A\,\mathrm{diag}(\lambda)^{-1} r_c + A\,\mathrm{diag}(\lambda)^{-1}\mathrm{diag}(x)\,r_d$$

$$\text{row1}: -I\Delta\lambda + A^T\Delta\nu = -r_d \qquad \Delta\lambda = -(-r_d - A^T\Delta\nu) = r_d + A^T\Delta\nu$$
$$\text{row2}: \mathrm{diag}(\lambda)\Delta x + \mathrm{diag}(x)\Delta\lambda = -r_c \qquad \Delta x = \mathrm{diag}(\lambda)^{-1}(-r_c - \mathrm{diag}(x)\Delta\lambda)$$