$$\min.\ f(x) \qquad \text{the optimal is } x^*$$
$$\text{s.t. } h_i(x) = 0$$

$$\mathcal{L}(x,v) = f(x) + \sum_i v_i h_i(x)$$
$$\mathcal{A}(x,v,c) = \mathcal{L}(x,v) + \frac{c}{2}\sum_i \|h_i(x)\|_2^2 = f(x) + \sum_i v_i h_i(x) + \frac{c}{2}\sum_i \|h_i(x)\|_2^2$$

optimality condition : $\begin{cases} h_i(x^*) = 0 \\ \nabla \mathcal{L}(x^*, v^*) = \nabla f(x^*) + \sum_i v_i^* \nabla h_i(x^*) \end{cases}$

both $\mathcal{L}(x,v)$ and $\mathcal{A}(x,v,c)$ is minimized at (x^*, v^*)

$$\nabla_x \mathcal{A}(x^*, v^*, c) = \nabla \mathcal{L}(x^*, v^*) + c\sum_i h_i(x) \cdot \nabla h_i(x^*) = 0$$

we want $\min_x \mathcal{A}(x,v,c)$ while keep $\nabla_x \mathcal{A}(x,v,c) = \nabla_x \mathcal{L}(x,v)$

$$x' = \arg\min_x \mathcal{A}(x,v,c)$$
$$\nabla_x \mathcal{A}(x',v,c) = \nabla f(x') + \sum_i v_i \nabla h_i(x') + c\sum_i h_i(x') \cdot \nabla h_i(x')$$
$$\nabla_x \mathcal{L}(x',v',c) = \nabla f(x') + \sum_i v_i' \nabla h_i(x')$$

$$v_i' = v_i + c \cdot h_i(x')$$

eg. least-norm solution

$$\min.\ x^T x \qquad \mathcal{L}(x,v) = x^T x + v^T(Ax-b)$$
$$\text{s.t. } Ax = b \qquad \mathcal{A}(x,v,c) = x^T x + v^T(Ax-b) + \frac{c}{2}\|Ax-b\|_2^2$$

$$x' = \arg\min_x \mathcal{A}(x,v,c)$$
$$\nabla_x \mathcal{A}(x',v,c) = 2x' + A^T v + c \cdot A^T(Ax-b) := 0$$
$$2x' + A^T v + c \cdot A^T A x - c \cdot A^T b = 0$$
$$(2I + c \cdot A^T A)x' = A^T(c \cdot b - v)$$

$$\nabla_x \mathcal{A}(x',v,c) = \nabla \mathcal{L}(x',v')$$
$$\nabla_x \mathcal{A}(x',v,c) = 2x' + A^T v + c \cdot A^T(Ax'-b)$$
$$\nabla_x \mathcal{L}(x',v') = 2x' + A^T v'$$
$$v' = v + c \cdot (Ax'-b)$$