- Schur complement

$X \in S^n \qquad X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$

$S = C - B^T A^{-1} B$ is the schur complement of X

$\det X = \det A \cdot \det S$

- Inverse of block matrix

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \qquad (\text{Assume } \det A \neq 0)$$

eliminal x from top block equation

$Ax + By = u$

$x = A^{-1}(u - By)$

$$\begin{aligned} v &= B^T A^{-1}(u - By) + Cy \\ &= B^T A^{-1} u + [C - B^T A^{-1} B] y \\ &= B^T A^{-1} u + Sy \end{aligned}$$

$y = S^{-1}(v - B^T A^{-1} u)$

substitute in equation 1

$Ax + By = u$

$$\begin{aligned} x &= A^{-1}(u - By) \\ &= A^{-1} u - A^{-1} B S^{-1}(v - B^T A^{-1} u) \end{aligned}$$

$x = (A^{-1} + A^{-1} B S^{-1} B^T A^{-1}) u - A^{-1} B S^{-1} v$

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1} B S^{-1} B^T A^{-1} & -A^{-1} B S^{-1} \\ -S^{-1} B^T A^{-1} & S^{-1} \end{bmatrix}$$

$X \succ 0 \iff A \succ 0$ and $S \succ 0$

if $A \succ 0$, $X \succeq 0 \iff S \succeq 0$