Lec 7: Optimization problems

- Generalized Inequality constraints

1. Convex problem with generalized inequality constraints

$$\begin{aligned} \min.\ & f_0(x) \\ \text{s.t.}\ & f_i(x) \preceq_{K_i} 0 \\ & Ax = b \end{aligned}$$

$f_0 : \mathbb{R}^n \to \mathbb{R}$ convex
$f_i : \mathbb{R}^n \to \mathbb{R}^{k_i}$ is K_i – Convex w.r.t. proper cone K_i

2. Conic form problems: special case with affine constraints

$$\begin{aligned} \min.\ & c^T x + d \\ \text{s.t.}\ & Fx + g \preceq_K 0 \\ & Ax = b \end{aligned}$$

extends linear programming ($K = \mathbb{R}^m_+$) to nonpolyhedra cones.

- Semidefinite programming (SDP)

$$\begin{aligned} \min.\ & c^T x \\ \text{s.t.}\ & x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \qquad (F_i, G \in S^n) \\ & Ax = b \end{aligned}$$

the inequality constraint is called Linear Matrix Inequality (LMI)

Includes problems with multiple LMI constraints:

$$\begin{cases} x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0 \\ x_1 \bar{F}_1 + \cdots + x_n \bar{F}_n + \bar{G} \preceq 0 \end{cases}$$

$\Downarrow$ equivalent to a single LMI

$$x_1 \begin{bmatrix} \hat{F}_1 & \\ & \bar{F}_1 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \bar{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \bar{G} \end{bmatrix} \preceq 0$$

Semidefinite programming is generalization of everything so far except geometric programming

1. LP as SDP

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax \le b \end{aligned} \quad \xRightarrow{\text{sdp embedding (what cvx does)}} \quad \begin{aligned} \min\ & c^T x \\ \text{s.t. } & \operatorname{diag}(Ax - b) \preceq 0 \end{aligned}$$

2. SOCP as SDP

$$\begin{aligned} \min.\ & f^T x \\ \text{s.t. } & \|A_i x + b_i\|_2 \le c_i^T x + d_i \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \min\ & f^T x \\ \text{s.t. } & \begin{bmatrix} (c_i^T x + d_i) I & (A_i x + b_i) \\ (A_i x + b_i)^T & (c_i^T x + d_i) \end{bmatrix} \succeq 0 \end{aligned}$$

$$\Updownarrow$$

$$\begin{cases} (c_i^T x + d_i) I \succeq 0 \\ \underbrace{(c_i^T x + d_i) - (A_i x + b_i)^T \left(\frac{1}{c_i^T x + d_i}\right) \cdot I^{-1} (A_i x + b_i)}_{\text{Schur complement}} \succeq 0 \end{cases}$$

$$\Downarrow$$

$$(c_i^T x + d_i)^2 \ge (A_i x + b_i)^T (A_i x + b_i) \quad \text{socp constraint}$$

eg. max eigenvalue minimization

$$\min.\ \lambda_{\max}(A(x)) \qquad A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \qquad A_i \in S^k$$

$$\Updownarrow$$

$$\begin{aligned} \min_{(x,t)}.\ & t \\ \text{s.t. } & A_0 + x_1 A_1 + \cdots + x_n A_n \preceq tI \quad \text{(affine in } x \text{ and } t) \end{aligned}$$

eg. matrix norm minimization

$$\min.\ \|A(x)\| = \left(\lambda_{max}\left(A(x)^T A(x)\right)\right)^{1/2}$$

$$\Updownarrow$$

$$\min.\ t$$

$$\text{s.t.}\ \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0 \iff \begin{cases} tI \succeq 0 \\ tI - A(x)^T (tI)^{-1} A(x) \succeq 0 \end{cases}$$

- vector optimization

1. general vector optimization:

$$\begin{aligned} &\min.\ (\text{w.r.t } K)\ f_0(x) \\ &\text{s.t.}\quad f_i(x) \le 0 \\ &\qquad\quad h_i(x) = 0 \end{aligned}$$

vector objective $f_0 : \mathbb{R}^n \to \mathbb{R}^q$, minimized w.r.t cone $K \in \mathbb{R}^q$

2. convex vector optimization problem

$$\begin{aligned} &\min.\ (\text{w.r.t } K)\ f_0(x) \\ &\text{s.t.}\quad f_i(x) \le 0 \\ &\qquad\quad Ax = b \end{aligned}$$

$f_0(x)$ is K-convex, f_i convex

3. optimal and pareto optimal points

set of achievable objective values $O = \{f_0(x) \mid x \text{ feasible}\}$

feasible x is optimal if $f_0(x)$ is a minimum value of O (all are more)
feasible x is pareto optimal if $f_0(x)$ is a minimal value of O (none is less)

$f(x^*)$ x^* is optimal

x^* is pareto optimal

4. multicriterion optimization

optimization problem with $K = \mathbb{R}^q_+$

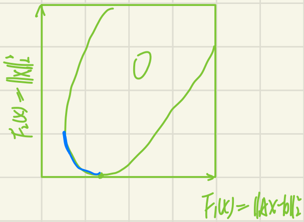
$f_0(x) = (F_1(x), \dots F_q(x))$ (want all F_i's to be small)

feasible x^* is optimal if : y feasible $\Leftrightarrow f_0(x^*) \preceq f_0(y)$

feasible x^{po} is pareto optimal if : y feasible $f_0(y) \preceq f_0(x^{po}) \Rightarrow y = x^{po}$

eg. regularized least-squares

min. (w.r.t $\mathbb{R}^2_+$) $(\|Ax-b\|_2^2, \|x\|_2^2)$



eg. risk return trade-off in portfolio optimization

$$\begin{aligned} \text{min.} \ (\text{w.r.t } \mathbb{R}^2_+) \quad & (-\bar{p}^T x, \ x^T \Sigma x) \\ \text{s.t} \quad & \mathbf{1}^T x = 1 \\ & x \succeq 0 \end{aligned}$$

$x \in \mathbb{R}^n$ is investment portfolio; x_i is fraction invested in asset i

$p \in \mathbb{R}^n$ is relative asset price changes;

modeled as a random var with mean $\bar{p}$, covariance Σ

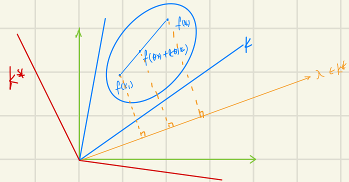
$\bar{p}^T x = E[r]$ is expected return ; $x^T \Sigma x = \text{var}(r)$ is return return variance

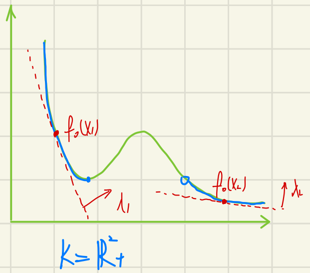
5. Scalarization

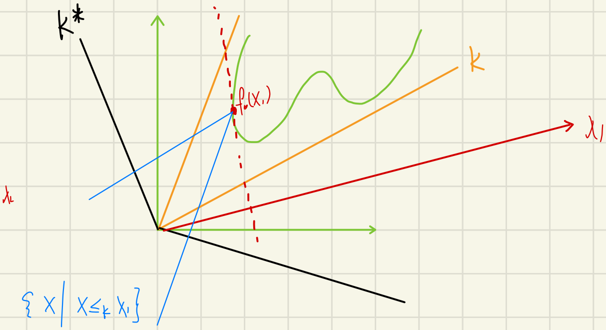
to find Pareto optimal points : choose $\lambda \succ_{K^*} 0$ and solve scalar problem

$$\begin{aligned} \text{min.} \ & \lambda^T f_0(x) \\ \text{s.t.} \ & f_i(x) \le 0 \\ & h_i(x) = 0 \end{aligned}$$

f_0 is K-convex if $\lambda^T f_0(x)$ is convex for all $\lambda \in K^*$




f₀(x₁)
f₀(x₂)
λ₁
λ₂
K = ℝ²₊


K*
K
λ₁
f₀(x₁)
{x | x ≤_K x₁}