Lec 6: Optimization problems

- Linear-fractional programming

$$\begin{aligned} \min.\ & f_0(x) \\ \text{s.t.}\ & Gx \preceq h \\ & Ax = b \end{aligned}$$

$$f_0(x) = \frac{c^Tx+d}{e^Tx+f} \qquad \mathrm{dom}\, f_0 = \{x \mid e^Tx+f>0\}$$

– a quasiconvex optimization problem; can be solved by bisection

$$f_0(x) = (c^Tx+d)/(e^Tx+f) \le t$$

$$\Updownarrow$$

$$c^Tx+d - t\cdot e^Tx - tf \le 0 ? \qquad \text{feasibility problem}$$

– equivalent to the LP

$$\begin{aligned} \min.\ & \frac{c^Tx+d}{e^Tx+f} \\ \text{s.t.}\ & Gx \preceq h \\ & Ax = b \end{aligned} \qquad \xRightarrow[\displaystyle z = \frac{1}{e^Tx+f}]{\displaystyle y = \frac{x}{e^Tx+f}} \qquad \left.\begin{aligned} \min\ & c^Ty + dz \\ \text{s.t.}\ & Gy \preceq hz \\ & Ay = bz \\ & e^Ty + fz = 1 \\ & z \ge 0 \end{aligned}\right\} \Rightarrow \text{a LP}$$

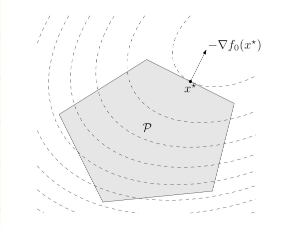
- Generalized Linear-fractional programming

$$f_0(x) = \max_i \frac{c_i^Tx+d_i}{e_i^Tx+f_i} \qquad \mathrm{dom}\, f = \{x \mid e_i^Tx+f_i>0\}$$

a quasiconvex problem; can be solved by bisection

- Quadratic programming

$$\begin{aligned} \min.\ & \tfrac{1}{2} x^TPx + q^Tx + r \qquad P \in S_+^n \\ \text{s.t.}\ & Gx \preceq h \\ & Ax = b \end{aligned}$$



minimizing a convex quadratic function over a polyhedron eg. least square

$$\min. \ \|Ax-b\|_2^2$$

analytical solution $x^* = A^+ b$ (A^+ is pseudo inverse)
can add linear constraints $l \le x \le u$ $\quad (x_1 \le x_2 \le \cdots \le x_n)$

eg. linear program with random cost

$$\begin{aligned} \min.\ & \bar{c}^T x + \gamma \cdot x^T \Sigma x \quad = E[c^T x] + \gamma \cdot \mathrm{var}(c^T x) \\ \text{s.t}\ & Gx \le h \\ & Ax = b \end{aligned}$$

- Quadratically constrained quadratic programming (QCQP)

$$\begin{aligned} \min.\ & 1/2\ x^T P_0 x + q_0^T x + r_0 \\ \text{s.t}\ & 1/2\ x^T P_i x + q_i^T x + r_i \le 0 \\ & Ax = b \end{aligned}$$

$P_i \in S_+^n$ (degenerate ellipsoid. eg. cylinder in $\mathbb{R}^3$)

- Second-order cone programming (SOCP)

$$\begin{aligned} \min.\ & f^T x \\ \text{s.t.}\ & \|A_i x + b_i\|_2 \le c_i^T x + d_i \\ & Fx = g \end{aligned}$$

— inequalities are called second-order cone constraints:
$(A_i x + b_i,\ c_i^T x + d_i) \in$ unit second-order cone in $\mathbb{R}^{n_i+1}$

$f_i(x) = \|A_i x + b_i\|_2 - c_i^T x - d_i$ is not differentiable.

· Robust linear programming

$$\min.\ c^Tx$$
$$\text{s.t. } a_i^Tx \le b_i \qquad (\text{uncertain } c, a_i, b_i)$$

1. deterministic approach via SOCP

chose an ellipsoid $\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \le 1\}$

$\Downarrow$

$$\min.\ c^Tx$$
$$\text{s.t. } a_i^Tx \le b_i \text{ for all } a_i \in \mathcal{E}_i$$

$\Updownarrow$

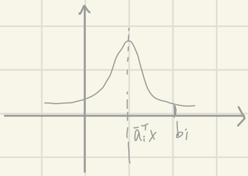
$$\sup_{\|u\|_2 \le 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$$

$$\left.\begin{array}{l}\min\ c^Tx \\ \text{s.t. } a_i^Tx + \|P_i^Tx\|_2 \le b_i\end{array}\right\} \text{ a second-order cone programming}$$

2. Stochastic approach via socp

assume a_i is Gaussian with mean $\bar{a}_i$, covariance Σ_i $\quad (a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i))$
a_i^Tx is Gaussian with mean $\bar{a}_i x$, variance $x^T \Sigma_i x$

$$\text{prob}(a_i^Tx \le b_i) = \Phi\left(\frac{b_i - \bar{a}_i^Tx}{\|\Sigma_i^{1/2} x\|_2}\right)$$



$$\frac{a_i^Tx}{\|\Sigma_i^{1/2} x\|_2} \sim \mathcal{N}(\bar{a}_i^Tx, 1)$$

$$\frac{b_i - a_i^Tx}{\|\Sigma_i^{1/2} x\|_2} \sim \mathcal{N}(b_i - a_i^Tx, 1)$$

$$\Phi(x) = 1/\sqrt{2\pi} \cdot \int_{-\infty}^{x} \exp(-t^2/2)\, dt \quad \text{is CDF of } \mathcal{N}(0,1)$$

$$\min\ c^Tx$$
$$\text{s.t. } \text{prob}(a_i^Tx \le b) \ge \eta$$

$\Updownarrow$ when $\eta \ge 1/2$, equivalent to socp

$$\min \quad c^T x$$
$$\text{s.t.} \quad \bar{a}_i^T x + \Phi^{-1}(\eta) \cdot \|\Sigma^{1/2} x\|_2 \le b_i$$

SOCP when $\Phi^{-1}(\eta) \ge 0 \Rightarrow \eta \ge 0.5$

- Geometric programming

monomial function:

$$f(x) = c \cdot x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n} \qquad \text{dom} f = \mathbb{R}_{++}^n \quad c>0 \quad a_i \text{ is any real number}$$

posynomial function: (sum of monomial)

$$f(x) = \sum_{k=1}^{K} c_k \cdot x_1^{a_{1k}} \cdot x_2^{a_{2k}} \cdots x_n^{a_{nk}}$$

geometric program:

$$\begin{aligned} \min. \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 1 \\ & h_i(x) = 1 \end{aligned}$$

f_i: posynomial

h_i: monomial

geometric program in convex form:

change variables to $y_i = \log x_i$, and take logarithm of cost and constraints

1. monomial $f(x) = c \cdot x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n}$ transforms to

$$\begin{aligned} \log f(x) &= \log c + a_1 \cdot \log x_1 + \cdots a_n \cdot \log x_n \\ &= a^T y + b \qquad (y_i = \log x_i \quad b = \log c) \end{aligned}$$

monomial $\Rightarrow$ affine

2. posynomial $f(x) = \sum_k c_k \cdot x_1^{a_{1k}} \cdot x_2^{a_{2k}} \cdots \cdot x_n^{a_{nk}}$ transforms to

$$\log f(x) = \log\left[\sum_k c_k \cdot \exp(y_1)^{a_{1k}} \cdots \cdot \exp(y_n)^{a_{nk}}\right]$$

$$= \log\left[\sum_k \exp \log c_k \cdot \exp(y_1)^{a_{1k}} \cdots \exp(y_n)^{a_{nk}}\right]$$

$$= \log\left[\sum_k \exp\left(\log c_k + a_{1k}\cdot y_1 + \cdots a_{nk}\cdot y_n\right)\right]$$

$$= \log\left[\sum_k \exp(a_k^T y + b_k)\right] \qquad (b_k = \log c_k)$$

posynomial $\Rightarrow$ log-sum-exp (affine) is cvx in y

3. geometric programming transforms to convex problem

$$\min. \quad \log\left(\sum_k \exp(a_{0k}^T y + b_{0k})\right)$$

$$\text{s.t.} \quad \log\left(\sum_k \exp(a_{ik}^T y + b_{ik})\right) \le 0$$

$$Gy + d = 0$$

eg. minimizing spectral radius of nonnegative matrix

Perron-Frobenius eigenvalues $\lambda_{pf}(A)$

- exists for (element wise) positive $A \in \mathbb{R}^{n\times n}$
- a real, positive eigenvalue of A, equal to spectral radius $\max_i |\lambda_i(A)|$
- determines asymptotic growth (decay) rate of A^k: $A^k \sim \lambda_{pf}(A)^k$ as $k \to +\infty$
- alternative characterization: $\lambda_{pf}(A) = \inf\{\lambda \mid Av \le \lambda v \text{ for some } v \succ 0\}$

A is a square, element-wise positive matrix; it's not symmetric, so it can have complex eigenvalues. The complex eigenvalue of largest magnitude (that's a spectral radius) is positive, and the associated eigenvectors can be chosen to be positive.

equivalent geometric programming.

$$\min. \ \lambda$$
$$\text{s.t.} \ \sum_{j=1}^{n} A(x)_{ij} \cdot v_j / (\lambda v_i) \le 1$$

variables λ, v, x