Lec 5: Optimization problems

· optimization problem in standard form

$$\begin{aligned} &\text{minimize} \quad f_0(x) \\ &\text{subject to.} \quad f_i(x) \le 0 \qquad i=1,\cdots,m \\ &\qquad\qquad\quad h_i(x) = 0 \qquad i=1,\cdots,p \end{aligned}$$

$x \in \mathbb{R}^n$ is the optimization vars
$f_0 : \mathbb{R}^n \to \mathbb{R}$ is the objective
$f_i : \mathbb{R}^n \to \mathbb{R}$ are the inequality constraints
$h_i : \mathbb{R}^n \to \mathbb{R}$ are the equality constraints

optimal value
$p^* = \inf\{ f_0(x) \mid f_i(x) \le 0 \ , h_i(x) = 0 \}$
$p^* = +\infty$ if the problem is infeasible
$p^* = -\infty$ if problem is unbounded below

· optimal and locally optimal points
x is feasible if $x \in \text{dom} f$ and it satisfies the constraints
a feasible x is optimal if $f_0(x) = p^*$; X_{opt} is the set of optimal points

x is locally optimal if there is a $R>0$ such that x is optimal for

$$\begin{aligned} &\min_z \quad f_0(z) \\ &\text{s.t.} \quad f_i(z) \le 0 \quad h_i(z) = 0 \\ &\qquad\ \ \|z - x\|_2 \le R \end{aligned}$$

· Implicit constraints
the standard form optimization problem has an implicit constraint

$$x \in D = \bigcap \text{dom} f_i \cap \bigcap \text{dom} h_i$$

D is the domain of the optimization problem

eg. $\min. \ -\sum \log(b_i - a_i^T x) \Rightarrow$ implicit constraints : $a_i^T x < b_i$

· Feasibility problem

$$\begin{aligned} &\text{find} \quad x \\ &\text{subject to} \quad f_i(x) \le 0 \\ &\qquad\qquad\quad\ h_i(x) = 0 \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} &\text{min.} \quad 0 \\ &\text{s.t.} \quad f_i(x) \le 0 \\ &\qquad\ \ h_i(x) = 0 \end{aligned}$$

$p^* = 0$ if problem is feasible ; any feasible x is optimal

$p^* = 0$ if problem is infeasible

· Convex optimization problem

$$\begin{aligned} &\text{minimize} \quad && f_0(x) \\ &\text{subject to} && f_i(x) \le 0 \quad && i = 1, \cdots, m \\ & && a_i^T x = b_i \quad && i = 1, \cdots, p \end{aligned}$$

$f_0, f_1, \cdots f_m$ is convex

equality constraints are affine

(problem is quasiconvex if f_0 is quasiconvex) ($f_1, \cdots f_m$ are convex)

feasible set of a convex optimization problem is convex

eg. $$\begin{aligned} &\text{min.} \quad f_0(x) = x_1^2 + x_2^2 \\ &\text{s.t.} \quad f_1(x) = x_1 / (1+x_2^2) \le 0 \\ &\qquad\ \ h_1(x) = (x_1 + x_2)^2 = 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} &\text{min.} \quad x_1^2 + x_2^2 \\ &\text{s.t.} \quad x_1 \le 0 \\ &\qquad\ \ x_1 + x_2 = 0 \end{aligned}$$

$\Downarrow$

not a cvx opt problem

f_1 is not cvx

h_1 is not affine

$\Downarrow$

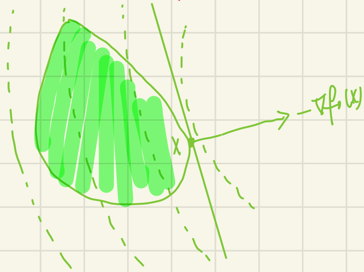
is a cvx opt problem

· local and global optima

any locally optimal point of a cvx opt problem is globally optimal

· optimality criterion for differentiable f_0

x is optimal $\iff$ x is feasible and $\nabla f_0(x)^T(y-x) \geq 0$ for all feasible y



if non-zero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set X at x

for all $x, y \in \mathrm{dom}\, f_0$

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y-x)$$

for y in the halfspace
$f(y) \geq f(x)$ since $\nabla f(x)^T(y-x) \geq 0$

eg. unconstrained problem

$\nabla f(x)^T(y-x) \geq 0$ for all (y)

$\Downarrow$

$g^T z = 0$ for all z

$\Downarrow$

$\nabla f(x) = 0$

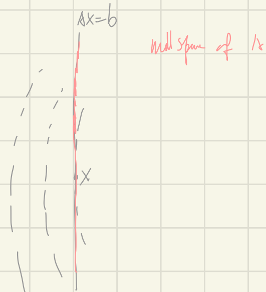
eg. equality constrained problem

min. $f_0(x)$

s.t. $Ax = b$

x is optimal iff there exists a v s.t.

$x \in \mathrm{dom}\, f_0 \quad Ax=b \quad \nabla f_0(x) + A^T v = 0$



$\nabla f_0(x)^T(z-x) \geq 0$ for all z. s.t. $Az = b$

$Ax = b \quad \therefore (z-x) \in \text{nullspace}(A)$

$\nabla f_0(x)$ has a non-negative with everything in nullspace (A)

$\therefore \nabla f_0(x) \perp \text{nullspace}(A)$

$\therefore \nabla f_0(x) \in \text{Range}(A^T)$

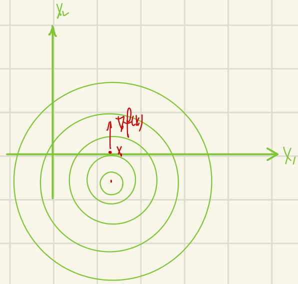
$(v \in \text{null}(A) \quad v \cdot \text{rowspace}(A) = 0)$

$\therefore \nabla f_0(x) = A^T \cdot u$

$v \in \text{nullspace}(A)$
$\Updownarrow$
$v \perp \text{rowspace}(A^T)$

eg minimization over non-negative orthant

$$\begin{aligned} \text{min. } & f_0(x) \\ \text{s.t } & x \geq 0 \end{aligned}$$



$\nabla f(x)^T(z-x) \geq 0$ for all $z \geq 0$

plug in $z=0$

$\nabla f(x)^T x \leq 0$

if $\nabla f(x)_i < 0$, make $z_i = t \quad t \to \infty$

$\nabla f(x)^T(z-x) = -\infty$

$\therefore \nabla f(x) \geq 0$

$\mathcal{L}(x,\lambda) = f(x) - \lambda^T x$

$\nabla \mathcal{L}_x = \nabla f(x) - \lambda = 0$

$\lambda = \nabla f(x)$

$\left\{ \begin{array}{l} \nabla f(x) \geq 0 \\ x \geq 0 \\ \nabla f(x)^T x \leq 0 \end{array} \right. \Rightarrow \nabla f(x)_i \cdot x_i = 0$ complementarity

- Equivalent convex problems

Two problems are equivalent if the solution of one is readily obtained from the solution of the other, and vice-versa

1. Eliminating Equality constraints

$$\begin{aligned} \text{min. } & f_0(x) \\ \text{s.t } & f_i(x) \leq 0 \quad i=1,\cdots,m \\ & Ax = b \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} \text{min. (over } z) \quad & f_0(Fz + x_0) \\ \text{s.t. } \quad & f_i(Fz + x_0) \leq 0 \quad i=1,\cdots,m \end{aligned}$$

(columns of F span the null space of A
x_0 is a particular solution of $Ax=b$)

2. Introducing equality constraints

$$\min. \quad f_0(A_0 x + b_0)$$
$$f_i(A_i x + b_i) \le 0 \qquad i=1,\cdots,m$$

$$\Updownarrow$$

$$\min_{(x,y)} \quad f_0(y_0)$$
$$f_i(y_i) \le 0$$
$$y_i = A_i x + b_i$$

3. introducing slack variables for linear inequalities

$$\min. \quad f_0(x)$$
$$\text{s.t.} \quad a_i^T x \le b_i$$

$$\Updownarrow$$

$$\min_{(x,s)} \quad f_0(x)$$
$$\text{s.t} \quad a_i^T x + s_i = b_i \qquad i=1,\cdots,m$$
$$s_i \ge 0 \qquad i=1,\cdots,m$$

4. epigraph form

$$\min_{(x,t)} \quad t$$ (minimizing a linear objective)
$$\text{s.t.} \quad f_0(x) - t \le 0$$
$$f_i(x) = 0$$
$$Ax = b$$

5. minimizing over some variables

$$\min \quad f_0(x_1, x_2)$$
$$\text{s.t.} \quad f_i(x_1) \le 0$$

$$\Updownarrow$$

dynamic programming preserves convexity of a problem

$$\min. \quad \tilde{f}_0(x_1)$$
$$\text{s.t.} \quad f_i(x_1) \le 0$$

$$\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$$

- Quasiconvex optimization

$$\begin{aligned} \min.\ & f_0(x) \\ \text{s.t.}\ & f_i(x) \le 0 \\ & Ax = b \end{aligned}$$

$f_0 : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex, f_i is convex

Can have locally optimal points thats not globally optimal

1. Convex representation of sublevel sets of f_0

   if f_0 is quasiconvex, there exists a family of function ϕ_t such that
   - $\phi_t(x)$ is convex in x for a fixed t
   - t-sublevel set of f_0 is 0-sublevel set of ϕ_t

     $f_0(x) \le t \iff \phi_t(x) \le 0$

eg.

$$f_0(x) = \frac{p(x)}{q(x)} \qquad \frac{\text{non-negative convex}}{\text{positive concave}} = \text{quasiconvex}$$

with $p(x)$ convex, $q(x)$ concave,
$p(x) \ge 0$ $q(x) > 0$ on dom f_0

$p(x)/q(x) \le t$ $(t \ge 0)$ set of x convex ?

$\Updownarrow$

$\underbrace{p(x) - t q(x)}_{\text{cvx}} \le 0$

take $\phi_t(x) = p(x) - t(q(x))$

$\Updownarrow$

$p(x)/q(x) \le t$ iff $p(x) - t(q(x)) \le 0$

2. quasiconvex optimization via convex feasibility problems

$$\begin{aligned} & \phi_t(x) \le 0 \\ & f_i(x) \le 0 \\ & Ax = b \end{aligned}$$

for fixed t, its a convex feasibility problem in x
if feasible $p^* \le t$; otherwise $p^* \ge t$

3. Bisection method for quasiconvex optimization

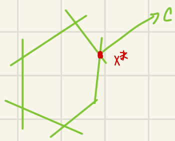
give $l \le p^*$, $u \ge p^*$, tol $\epsilon > 0$
repeat
$\quad t = (u+l)/2$
$\quad$ solve feasibility problem
$\quad$ if feasible $u = t$; otherwise $l = t$
until $(u-l)/\epsilon$

exactly $\log_2((u-l)/\epsilon)$ iterations

- Linear program (LP)

$$\begin{aligned} \min.\ & c^T x + d \\ s.t.\ & Gx \preceq h \\ & Ax = b \end{aligned}$$

convex optimization problem with affine constraints and affine objectives
feasible set is a polyhedron

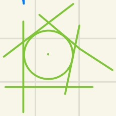


eg. piecewise-linear minimization

$$\left.\begin{aligned} \min.\ & \max_i (a_i^T x + b) \\ & \Updownarrow \\ \min.\ & t \\ s.t.\ & a_i^T x + b \le t \end{aligned}\right\} \text{LP}$$

eg. chebyshev center for polyhedron

chebyshev center of $P = \{x \mid a_i^T x \le b_i\}$
is the center of the largest inscribed ball $B = \{x_c + u \mid \|u\|_2 \le r\}$



$a_i^T x \leq b_i$ for all $x \in B$ iff

$$\sup\{a_i^T(x_c+u) \mid \|u\|_2 \leq r\} = a_i^T x_c + r \cdot \|a_i\|_2 \leq b_i$$

$$\left.\begin{array}{ll} \text{maxi.} & r \\ \text{s.t.} & a_i^T x + r \cdot \|a_i\|_2 \leq b_i \end{array}\right\} \text{linear programming}$$