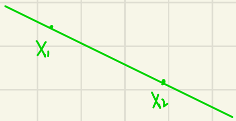
Lec 2 : Convex Sets

- Affine set

all points on a line through x_1 x_2

$$x = \theta x_1 + (1-\theta) x_2$$



eg. solution set of $\{x \mid Ax = b\}$

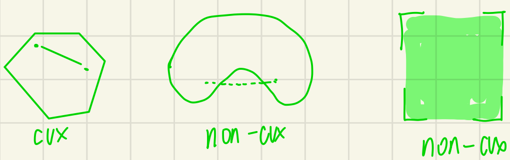
- Convex set

line segment between x_1 x_2 : $x = \theta x_1 + (1-\theta) x_2 \qquad \theta \in [0,1]$

convex set : contains line segment between any 2 points in the set

$$x_1 \; x_2 \in C \quad 0 \le \theta \le 1 \Rightarrow \theta x_1 + (1-\theta) x_2 \in C$$

eg. convex sets.

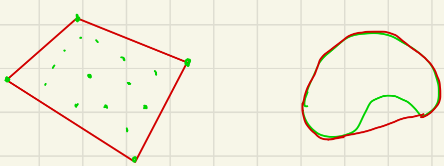


- Convex combination and convex hull

convex combination : $x = \theta_1 x_1 + \cdots + \theta_k x_k \quad \sum \theta_i = 1 \quad \theta_i \ge 0$

convex hull :

conv S : set of all convex combination of points in S



convex hull of a cvx set is the set itself

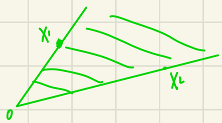
- Convex cone

conic (nonnegative) combination of x_1 and x_2:

$$x = \theta_1 x_1 + \theta_2 x_2 \quad \theta_1 \geq 0 \quad \theta_2 \geq 0$$

convex cone: set that contains all conic combination of points in the set

any cvx cone has to be a cvx set



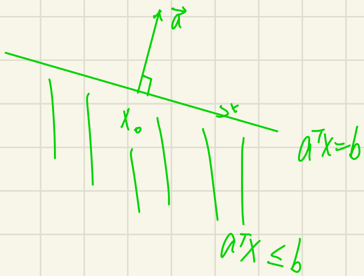
- Hyperplanes and halfspaces

hyperplane: $\{x \mid a^T x = b\}$ $a \neq 0$

halfspace: $\{x \mid a^T x \leq b\}$ $a \neq 0$

hyperplane is cvx and affine

halfspace is cvx



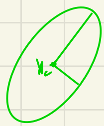
- Euclidean balls and ellipsoids

Euclideal balls with center x_c and radius r

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + r \cdot u \mid \|u\|_2 \leq 1\}$$

Ellipsoids:

$$\{x \mid (x - x_c)^T \cdot P^{-1} (x - x_c) \leq 1\}$$ with $P \in S^n_{++}$ (P symmetric, pd)



$$P = U \Lambda U^{-1} = U \Lambda U^T$$

$$(x - x_c)^T \cdot U \Lambda^{-1} \cdot U^T (x - x_c)$$

$$= v^T \Lambda^{-1} v$$

$$v = U^T (x - x_c) = U^{-1} (x - x_c)$$

where U is the normal axis

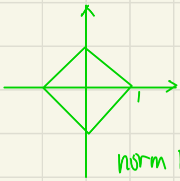
other representation: $\{x_c + Au \mid \|u\|_2 \leq 1\}$

- Norm balls and norm cones

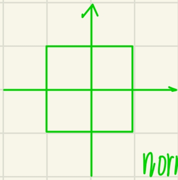
  norm: any function that satisfies
  - $\|x\| \geq 0$, $\|x\| = 0$ iff $x = 0$
  - $\|tx\| = |t| \cdot \|x\|$ for $t \in \mathbb{R}$
  - $\|x+y\| \leq \|x\| + \|y\|$

  norm ball with center x_c and radius r

  $$\{ x \mid \|x - x_c\| \leq r \}$$



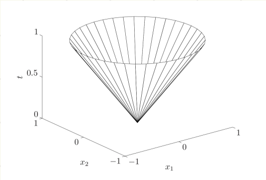
norm ball in $\|\ \|_1$



norm ball in inf

  norm cone.

  $$\{ (x, t) \mid \|x\| \leq t \}$$



  Euclidean norm cone is called second-order cone

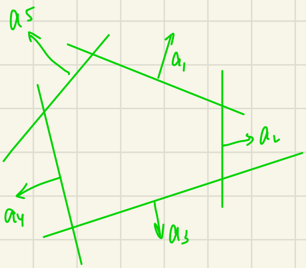
  norm balls and norm cones are cvx

- Polyhedra

  solution set of finitely many linear inequalities and equalities

  $$Ax \leq b \quad Cx = d$$

  polyhedra is intersection of finite number of halfspaces and hyperplanes



- Positive Semi-definite Cone

S^n is set of symmetric $n \times n$ matrices

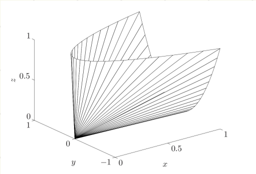
S^n_+ is set of symmetric semi-def matrices ($X \in S^n_+$ $z^T X z \geq 0$)

$X_1 \in S^n_+$ $X_2 \in S^n_+$ $\theta_1 X_1 + \theta_2 X_2 \in S^n_+$ for $\theta_1 \geq 0$ $\theta_2 \geq 0$

S^n_+ is a cvx cone

S^n_{++} set of symmetric def matrices

$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S^2_+$



Operations that preserve convexity

- Intersection

intersection of any number of cvx set is a cvx set

- Affine functions

suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ is a affine function $f(x) = Ax + b$

the image of a cvx set under f is cvx

$S \subseteq \mathbb{R}^n$ cvx $\Rightarrow$ $f(S) = \{ f(x) \mid x \in S \}$ is cvx

the inverse image $f^{-1}(C)$ of a cvx set under f is cvx

$C \subseteq \mathbb{R}^m$ cvx $\Rightarrow$ $f^{-1}(C) = \{ x \in \mathbb{R}^n \mid f(x) \in C \}$ cvx

eg. scaling. translation, projection

solution set of linear matrix inequalities $\{ x \mid x_1 A_1 + \cdots + x_n A_n \leq B \}$ $A_i, B \in S^p$

$f(x) = B - A(x)$

$= B - x_1 A_1 - \cdots - x_n A_n$

$A_i, B \in S^p_+$ is a cvx set

$f^{-1}(S^p_+) = \{ x \in \mathbb{R}^n \mid B - A(x) \in S^p_+ \}$

$\{ x \mid A(x) \leq B \}$ is cvx

eg. hyperbolic cone

$A = \{(z,u) \mid \|z\| \le u, u \ge 0\}$ is a cvx cone

$$f(x) = \begin{bmatrix} P^{1/2} x \\ c^T x \end{bmatrix}$$

$$\Downarrow$$

$$\{x \mid x^T P x \le (c^T x)^2, c^T x \ge 0\}$$

$$= \{x \mid f(x) = A\}$$

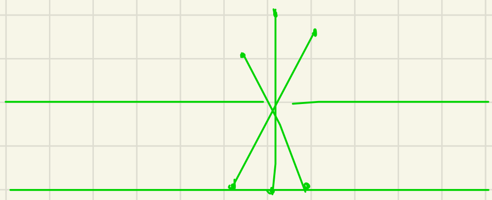
f is a affine mapping
$\{x \mid f(x) = A\}$ A is a cvx set
$\Big\}$ $\{x \mid f(x) = A\}$ is a cvx set

· perspective and linear-fractional function

perspective function $P: \mathbb{R}^{n+1} \to \mathbb{R}^n$

$P(x,t) = x/t$ dom $P = \{(x,t) \mid t > 0\}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

images and inverse images of convex sets under perspective functions are convex



linear fractional function (generalization of perspective function)

$$f(x) = \frac{Ax+b}{c^T x + d} \quad \text{dom} f = \{x \mid c^T x + d > 0\} \quad \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}$$

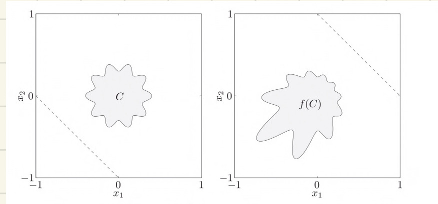
images and inverse images of convex sets under linear-fractional functions are cvx functions

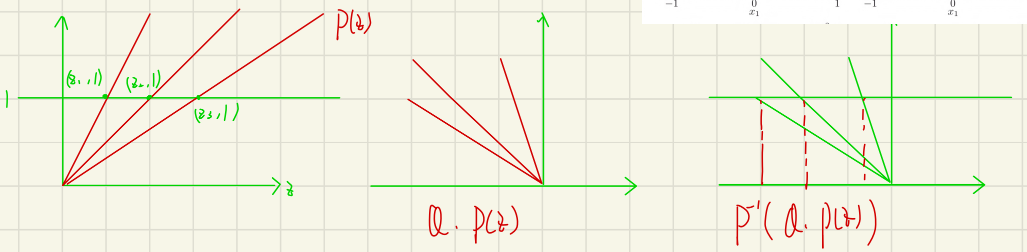
projective interpretation

$$Q = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix} \in \mathbb{R}^{(m+1)\times(n+1)}$$

$Q\cdot\begin{bmatrix}x\\1\end{bmatrix} = [Ax+b \quad c^Tx+d]^T$

$P(z)=\{t\cdot(z,1)\mid t>0\}\in\mathbb{R}^{n+1}$





Generalized Inequalities

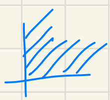
· Proper Cone

A cvx cone $K\subseteq\mathbb{R}^n$ is a proper cone if
- K is closed (contains its boundary)
- K is solid (nonempty interior)
- K is pointed (contains no line)

eg.

nonnegative orthant $K=\mathbb{R}^n_+$

psd cone $K=S^n_+$



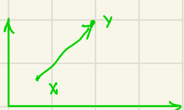
· Generalized Inequality

generalized inequality is defined by a proper cone K

$x\preceq_K y \Leftrightarrow y-x\in K$ $\qquad$ $x\prec_K y \Leftrightarrow y-x\in \text{int}(K)$

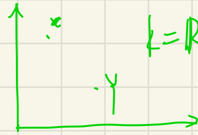
eg1. component-wise inequality $(K=\mathbb{R}^n_+)$

$x\preceq_K y \Leftrightarrow y-x\in\mathbb{R}^n_+$



eg2. matrix inequality $(K=S^n_+)$

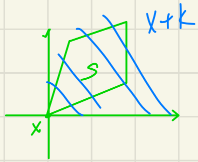
$X\preceq_{S^n_+} Y \Leftrightarrow Y-X$ is psd not a total ordering $x \not\le_K y$ and $x \not\ge_K y$



$K=\mathbb{R}^n_+$

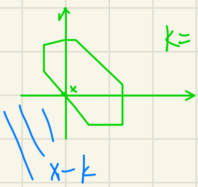
· minimum and minimal elements

$x \in S$ is the minimum element of S w.r.t K if
$y \in S \Rightarrow x \le_K y$



$K=\mathbb{R}^2_+$ all are more

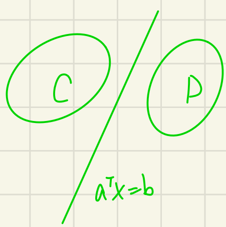
$x \in S$ is the minimal element of S w.r.t K if
$y \in S$ $y \le_K x \Rightarrow y = x$



$K=\mathbb{R}^2_+$ none is less

· Separating hyperplane theorem

if C and D are disjoint cvx sets, then there exists $a \ne 0$
$a^Tx \le b$ for $x \in C$ and $a^Tx \ge b$ for $x \in D$

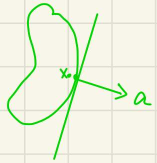


- Supporting hyperplane theorem.

supporting hyperplane to set C at boundary point x_0

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

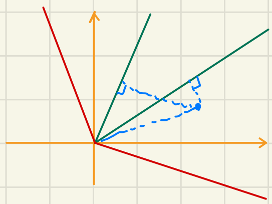


if C is a cvx set, then there exists a supporting hyperplane at every boundary point

- Dual Cone and Generalized Inequalities

dual cone of a cone K

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$



eg $K = \mathbb{R}^n_+$ $\quad K^* = \mathbb{R}^n_+$

eg $K = S^n_+$ $\quad K^* = S^n_+$ (inner product of 2 mats $\langle X, Y\rangle = \mathrm{Tr}(XY^T)$)

suppose $Y \notin S^n_+$, then there exists some $q \in \mathbb{R}^n$

$$q^T Y q = \mathrm{tr}(qq^T Y) < 0$$

$X = qq^T \in S^n_+$; $\mathrm{tr}(XY) < 0$; $Y \notin (S^n_+)^*$

$\therefore$ if $Y \notin S^n_+$; then $Y \notin (S^n_+)^*$

Suppose $X, Y \in S^n_+$

$$X = Q \Lambda Q^T = \sum_{i=1}^n \lambda_i q_i q_i^T \quad (\lambda_i \geq 0)$$

$$\mathrm{tr}(YX) = \mathrm{tr}\left(Y \cdot \sum_{i=1}^n \lambda_i q_i q_i^T\right) = \sum_{i=1}^n \lambda_i \cdot q_i^T Y q_i \geq 0$$

$\therefore Y \in (S^n_+)^*$

eg. $K = \{(x,t) \mid \|x\|_2 \le t\}$ $K^* = K$ (euclidean-norm cone)

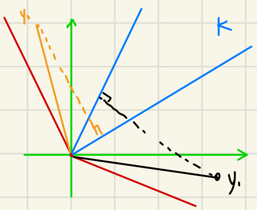
eg. $K = \{(x,t) \mid \|x\|_1 \le t\}$ $K^* = \{(x,t) \mid \|x\|_\infty \le t\}$

dual cone of 1-norm cone is inf-norm cone

$(K^*)^* = K$ if K is a proper cone

dual cone of proper cones are proper, hence define generalized inequality

$y \succeq_{K^*} 0 \iff y^T x \ge 0$ for all $x \succeq_K 0$



- Minimum and minimal elements via dual inequalities

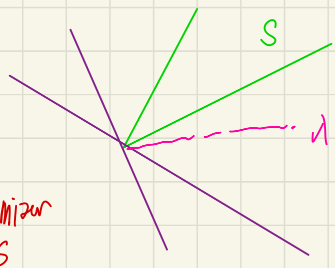
minimum element w.r.t. K (all are more)

x is the minimum element of S

$\Updownarrow$

for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over S

(project S on λ)

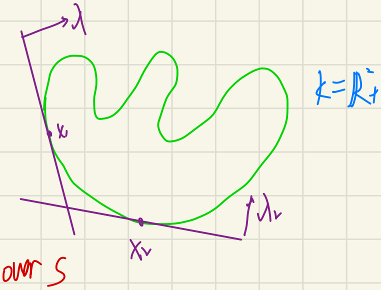


minimal element w.r.t K (none is less)

x is a minimal

$\Updownarrow$

for some $\lambda \succ_{K^*} 0$, x minimizes $\lambda^T z$ over S



if x is a minimal of a cvx set S, then there exists a $\lambda \succeq_{K^*} 0$, $\lambda \ne 0$ such that x minimizes $\lambda^T z$ over S

- optimal production frontier

minimal w.r.t $\mathbb{R}^2_+$