- Logarithmic barrier

$$\begin{aligned} \min.\ & f_0(x) + \sum_i I(f_i(x)) \\ \text{s.t.}\ & Ax=b \end{aligned} \Rightarrow \left.\begin{aligned} \min.\ & f_0(x) - \frac{1}{t}\cdot\sum_i \log(-f_i(x)) \\ \text{s.t.}\ & Ax=b \end{aligned}\right\} \begin{aligned} &\text{smooth} \\ &\text{solved by newton} \end{aligned}$$

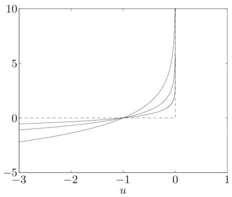

Figure 11.1 The dashed lines show the function $I_-(u)$, and the solid curves show $\widehat{I}_-(u) = -(1/t)\log(-u)$, for $t = 0.5,\ 1,\ 2$. The curve for $t = 2$ gives the best approximation.

$$\phi(x) = -\sum_i \log(-f_i(x)) \qquad \text{dom} f = \{x \mid f_i(x) < 0\}$$

$$= -\underbrace{\log \prod_i (-f_i(x))}_{} \qquad \text{slacks}$$

$$\nabla\phi(x) = \sum_i \frac{1}{-f_i(x)} \cdot \nabla f_i(x)$$

$$\nabla^2\phi(x) = \sum_i \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_i \frac{1}{-f_i(x)} \nabla^2 f(x)$$

- Central path

for $t>0$, define $x^*(t)$ as solution of

$$\begin{aligned} \min.\ & t f_0(x) + \phi(x) \\ \text{s.t.}\ & Ax=b \end{aligned} \qquad \left(\begin{aligned} \min.\ & f_0(x) - 1/t \sum_i \log(-f_i(x)) \\ \text{s.t.}\ & Ax=b \end{aligned}\right)$$

central path: $\{x^*(t) \mid t>0\}$

as $t \to +\text{inf}$ $\quad x^*(t) \to x^*$ for the original problem

eg. $\quad \min.\ c^T x$

$\quad\quad$ s.t. $a_i^T x \le b_i$

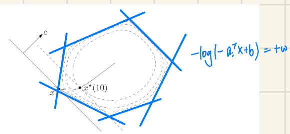

Figure 11.2 Central path for an LP with $n = 2$ and $m = 6$. The dashed curves show three contour lines of the logarithmic barrier function ϕ. The central path converges to the optimal point $x^\star$ as $t \to \infty$. Also shown is the point on the central path with $t = 10$. The optimality condition (11.9) at this point can be verified geometrically: The line $c^T x = c^T x^\star(10)$ is tangent to the contour line of ϕ through $x^\star(10)$.

• Dual points on central path

$$\min.\ t f_0(x) - \sum_i \log(-f_i(x))$$
$$\text{s.t. } Ax = b$$

$$\mathcal{L}(x,v) = t\cdot f_0(x) - \sum_i \log(-f_i(x)) + v^T(Ax-b)$$

optimality condition
$$\begin{cases} \nabla \mathcal{L}_x(x,v) = t\cdot\nabla f_0(x) + \sum_i \frac{1}{-f_i(x)}\cdot\nabla f_i(x) + A^T v := 0 \\ Ax - b = 0 \end{cases}$$

$x^*(t)$ is optimal for the barrier problem

$$\nabla f_0(x^*(t)) + \sum_i \underbrace{\frac{1}{-t\cdot f_i(x^*(t))}}_{\lambda_i^*(t)}\cdot\nabla f_i(x^*(t)) + A^T\underbrace{(v/t)}_{v^*(t)} = 0$$

$x^*(t)$ minimize the Lagrangian

$$\mathcal{L}(x, \lambda^*(t), v^*(t)) = f_0(x) + \sum_i \lambda_i^*(t)\cdot f_i(x^*(t)) + v^*(t)^T(Ax-b)$$

where $\lambda_i^*(t) = 1/[-t\cdot f_i(x^*(t))]$ $\quad v^*(t) = v/t$ $\quad$ dual feasible $\Rightarrow$ lower bound

$$\min.\ f_0(x)$$
$$\text{s.t. } f_i(x) \le 0$$
$$Ax = b$$

$$\mathcal{L}(x,\lambda,v) = f_0(x) + \sum_i \lambda_i f_i(x) + v^T(Ax-b)$$
$$g(\lambda,v) = \inf_x \mathcal{L}(x,\lambda,v)$$

$$\left.\begin{array}{l}\min.\ t f_0(x) - \sum_i \log(-f_i(x)) \\ \text{s.t. } Ax=b\end{array}\right\}$$ Solving this problem gives a primal-dual pair, thus a lower bound of the original problem

$$\begin{aligned} p^* &\ge g(\lambda^*(t), v^*(t)) \\ &= \mathcal{L}(x^*(t), \lambda^*(t), v^*(t)) \\ &= f_0(x^*(t)) + \sum_i \frac{1}{-t f_i(x^*(t))}\cdot f_i(x^*(t)) + v^*(t)(Ax^*(t) - b) \\ &= f_0(x^*(t)) - m/t \end{aligned}$$

if $x^*(t)$ minimizes $\left\{\begin{array}{l}\min.\ t f_0(x) - \sum_i \log(-f_i(x)) \\ \text{s.t. } Ax=b\end{array}\right\}$, then $f_0(x^*(t)) - m/t \le p^* \le f_0(x^*(t))$

$f_0(x^*(t))$ is no more than m/t subopt

- Interpretation via KKT conditions

$$\nabla f_0(x^*(t)) + \sum_i \underbrace{\frac{1}{-t \cdot f_i(x^*(t))}}_{\lambda_i^*(t)} \cdot \nabla f_i(x^*(t)) + A^T \underbrace{(v/t)}_{v^*(t)} = 0$$

$x = x^*(t)$, $\lambda = \lambda^*(t)$, $v = v^*(t)$ satisfies:

1. primal constraints: $f_i(x) \le 0$ $\quad Ax = b$ (actually strictly feasible i.e. $f_i(x) < 0$)
2. dual constraints: $\lambda \ge 0$ (actually $\lambda > 0$)
3. approximate complementary slackness: $\lambda_i f_i(x) = -1/t$
4. gradient w.r.t x vanishes:
$$\nabla f_0(x) + \sum_i \lambda_i \nabla f_i(x) + A^T v = 0$$

$x = x^*(t)$ $\lambda = \lambda^*(t)$ $v = v^*(t)$ satisfies 3 of 4 kkt conditions of the original problems
1 is approximated

- Force field interpretation

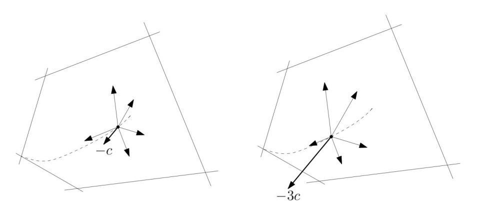

Figure 11.3 *Force field interpretation of central path.* The central path is shown as the dashed curve. The two points $x^\star(1)$ and $x^\star(3)$ are shown as dots in the left and right plots, respectively. The objective force, which is equal to $-c$ and $-3c$, respectively, is shown as a heavy arrow. The other arrows represent the constraint forces, which are given by an inverse-distance law. As the strength of the objective force varies, the equilibrium position of the particle traces out the central path.

- Barrier method

given strictly feasible x, $t = t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$ $\quad$ ($t = 2, 5, 10, 20 \cdots$)

repeat {

Centering step: computing $x^*(t)$ by minimizing $t f_0(x) + \phi$ $\quad Ax = b$
update $x := x^*(t)$
Stopping Criterion: quit if $m/t < \epsilon$
increase $t = \mu \cdot t$

}

- Convergence analysis

number of outer iterations: exactly

$$\left\lceil \frac{\log(m/\epsilon t^{(0)})}{\log \mu} \right\rceil \quad (\text{big if } \mu \text{ small})$$

Centering problem

$$\min.\ t f_0(x) + \phi(x)$$

if f_0 and log barrier is self concordant, then $tf_0 + \phi$ is s.c for $t \geq 1$

$$\frac{f(x^*(t-1) - p^*(t))}{\gamma} + \log\log 1/\epsilon$$

- Newton step for modified KKT equations

$$\begin{aligned} &\min.\ t \cdot f_0(x) + \phi(x) = f_0(x) + \left[-\sum_i \log(-f_i(x))\right] = f(x) \\ &\text{s.t.}\quad Ax = b \end{aligned}$$

optimality condition
$$\begin{cases} \nabla f(x+\Delta x) + A^T v \approx \nabla f(x) + \nabla^2 f(x)\Delta x + A^T v = 0 \\ A(x+\Delta x) = b \end{cases}$$

$$\nabla f(x) = t\nabla f_0(x) + \nabla\phi(x) = t\cdot\nabla f_0(x) + \sum_i \frac{-1}{f_i(x)} \cdot \nabla f_i(x)$$

$$\nabla^2 f(x) = t\nabla^2 f_0(x) + \nabla^2\phi(x) = t\cdot\nabla^2 f_0(x) + \sum_i \frac{-1}{f_i(x)} \nabla^2 f_i(x) + \sum_i \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T$$

$$\begin{bmatrix} t\cdot\nabla^2 f_0(x) + \nabla^2\phi(x) & A^T \\ A & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ v \end{bmatrix} = \begin{bmatrix} -t\nabla f_0(x) + \nabla\phi(x) \\ 0 \end{bmatrix}$$

- Feasibility and Phase 1 method.

feasibility problem: find x such that
$f_i(x) < 0 \quad Ax = b$

1. basic phase 1 method

$$\begin{aligned} &\text{min. (over } x, s) && s \\ &\text{s.t.} && f_i(x) - s \le 0 \\ & && Ax = b \end{aligned}$$

if x, s feasible, with $s < 0$, then x is strictly feasible
(if s get negative, then quit)

if s^* is positive, the the original problem is infeasible
if s^* is 0 and attained, then the problem is (not strictly) feasible

push s to the left, if it stops going left
a whole bunch of constraints are violated by s

2. Sum of infeasibilities phase 1 method

$$\begin{aligned} &\min && 1^T s \\ &\text{s.t.} && s \ge 0 \quad f_i(x) \le s_i \\ & && Ax = b \end{aligned}$$

for a infeasibility problem, produce a solution
that satisfies as many constraints as possible

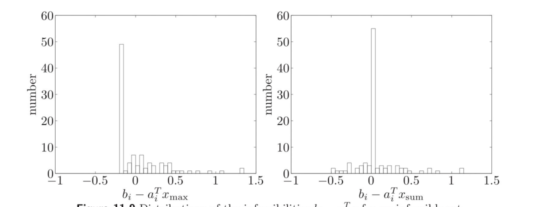

Figure 11.9 Distributions of the infeasibilities $b_i - a_i^T x$ for an infeasible set of 100 inequalities $a_i^T x \le b_i$, with 50 variables. The vector $x_{\max}$ used in the left plot was obtained by the basic phase I algorithm. It satisfies 39 of the 100 inequalities. In the right plot the vector x_{sum} was obtained by minimizing the sum of the infeasibilities. This vector satisfies 79 of the 100 inequalities.