- LU factorization

every nonsingular matrix A can be factored as

$$A = PLU$$

with P a permutation matrix, L lower triangular, U upper triangular

cost: $(2/3)n^3$ flops

Solving linear equations by LU factorization $Ax=b$

Factor	$A=PLU$	$2/3\ n^3$ flops
permutation:	$Pz_1=b$	0 flops
Forward sub:	$Lz_2=z_1$	n^2 flops
Backward sub:	$Ux=z_2$	n^2 flops

- Sparse LU factorization

$$A = P_1 L U P_2$$

(both row and column permutation)

P_1 and P_2 chosen heuristically to yield sparse L, U

- Cholesky factorization

every positive definite A can be factored as

$$A = LL^T$$

with L lower triangular $1/3\ n^3$ flops

$$\begin{bmatrix} A_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} L_{11}^T & L_{21}^T \\ 0 & L_{22}^T \end{bmatrix}$$

$A_{11} = L_{11} L_{11}^T \qquad A_{21} = L_{21} L_{11}^T$

$A_{22} = L_{21} L_{21}^T + L_{22} L_{22}^T$

Solving linear equations by Cholesky factorization $Ax=b$ $A \in S_{++}^n$

factorization:	$A=LL^T$	$1/3\ n^3$ flops
forward sub:	$Lz_1=b$	n^2 flops
backward sub:	$L^Tx=z_1$	n^2 flops

Cholesky is the best way to check if a matrix is p.d.

- Sparse Cholesky factorization

$$A = PLL^TP^T \qquad PAP^T = LL^T$$

- LDL^T factorization

every nonsingular symmetric A can be factored as

$$A = PLDL^TP^T \qquad 1/3\ n^3 \text{ flops}$$

P: permutation L: lower diag U: upper diag D: block diag

- Equations with structured sub-blocks

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

if A_{11} is nonsingular : $(A_{22} - A_{21} A_{11}^{-1} A_{12}) x_2 = b_2 - A_{21} A_{11}^{-1} b_1$

Schur complement (Solve arrow-shaped system)

Solving linear equations by block elimination (A_{11} non singular)

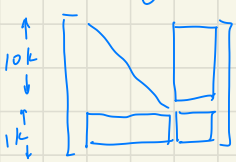
Form $A_{11}^{-1} A_{12}$ and $A_{11}^{-1} b_1$ $A_{11} Y_1 = A_{12}$ $A_{11} Y_2 = b_1$ factor A_{11} once

Form $S = A_{22} - A_{21} A_{11}^{-1} A_{12}$ and $\tilde{b} = b_2 - A_{21} A_{11}^{-1} b_1$

Determine x_2 : $S x_2 = \tilde{b}$

Determin x_1 : $A_{11} x_1 + A_{12} x_2 = b_1$ A_{11} already factored in step 1.

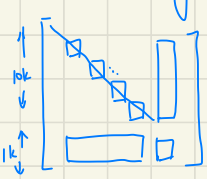
eg. A_{11} is diagonal



$A_{11}^{-1} A_{12}$ and A_{11}^{-1} is super cheap

$A_{22} - A_{21} A_{11}^{-1} A_{12}$ dominates the flop count

eg. A_{11} is block diagonal



form $A_{11}^{-1} A_{12}$ $A_{11} V = A_{12}$

factor each block of A_{11}

- Structured matrix plus low-rank.

$(A + BC) x = b$

$A \in \mathbb{R}^{n \times n}$ $B \in \mathbb{R}^{n \times p}$ $C \in \mathbb{R}^{p \times n}$

A has structure ($Ax = b$ easy to solve)

$$\begin{bmatrix} A & B \\ C & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \quad \begin{cases} Ax + By = b \\ Cx = y \end{cases}$$

$$\begin{cases} (-I - CA^{-1}B) y = -CA^{-1} b \\ Ax = b - By \end{cases}$$

The matrix inversion lemma: if A and $A+BC$ non singular

$$(A+BC)^{-1} = A^{-1} + A^{-1}B(I + CA^{-1}B)^{-1}CA^{-1}$$

eg. diagonal-plus-low-rank

$$(D + pq^T)x = b$$