Lec 12 : Application, geometric problems

- Minimum volume ellipsoid around a set

Lowner-John ellipsoid of a set C: minimum volume ellipsoid $\mathcal{E}$ s.t. $C \subseteq \mathcal{E}$

parametrize $\mathcal{E}$ as $\mathcal{E} = \{v \mid \|Av+b\|_2 \le 1\}$ $(A \in S^n_{++})$

(inverse image of a unit ball under a affine mapping)

(this representation of ellipsoid make the problem convex)

can assume A is positive definite without loss of generality

$$\{x \mid \|Ax-b\|_2 \le 1\}$$
$$= \{x \mid \|U\Lambda V^T x - b\|_2 \le 1\}$$
$$= \{x \mid \|VU^T(U\Lambda V^T x - b)\|_2 \le 1\}$$

multiplying a vector on the left by an orthogonal matrix doesn't change the norm

$$= \{x \mid \|V\Lambda V^T x - VU^T b\|_2 \le 1\}$$
$$= \{x \mid \|\tilde{A}x - \tilde{b}\|_2 \le 1\}$$

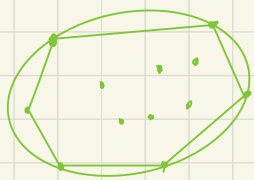
We only need to assume that A is nonsigular

volume of $\mathcal{E}$ is proportional to $\det(A^{-1})$

$$\begin{aligned} \text{min. (over } A, b) \quad & \log\det(A^{-1}) \\ \text{s.t.} \quad & \|Av+b\|_2 \le 1 \quad \text{for all } v \in \mathcal{E} \end{aligned}$$

(hard to evaluate the constraints, since it needs all $v \in \mathcal{E}$)

finite set $C = \{x_1, \dots, x_m\}$



$$\begin{aligned} \text{min.} \quad & \log\det(A^{-1}) \\ \text{s.t.} \quad & \|Av+b\|_2 \le 1 \quad i = 1, 2, \dots, m \end{aligned}$$

also gives the lowner-John ellipsoid for polyhedron Convex Hull $\{x_1, \dots, x_m\}$

eg. outlier detection

select the outliers in $\{v_1, \dots v_m\}$

fit the Lowner-John ellipsoid of $\{v_1 \cdots v_m\}$

the points right on the boundry are candidates for outliers

remove those points and redo

removing values of v with large norm is not scaling invariant

minimum volume ellipsoid is affine-invariant

- maximum volume inscribe ellipsoid

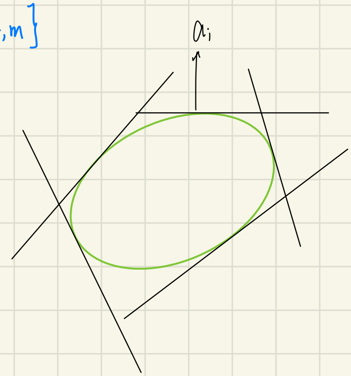
  maximum volume ellipsoid $\mathcal{E}$ inside a convex set $C \subseteq \mathbb{R}^n$

  parametrize $\mathcal{E}$ as $\mathcal{E} = \{ Bu + d \mid \|u\|_2 \le 1 \}$ image of a unit ball under a affine mapping

  volume of $\mathcal{E}$ is proportional to B

$$\max \ \log\det B$$
$$\text{s.t.} \quad Bu + d \in C \quad \forall \|u\|_2 \le 1$$ (a convex constraint if c is convex)

when C is a polyhedron $\{x \mid a_i^T x \le b_i \ \ i=1,\cdots,m\}$



$\mathcal{H} = \{x \mid f^T x \le g\}$

$Bu + d \in \mathcal{H} \quad \forall \|u\|_2 \le 1$??

$f^T(Bu+d) \le g \quad \forall \|u\|_2 \le 1$??

$(B^T f)^T u + f^T d \le g \quad \forall \|u\| \le 1$??

$\Downarrow$

$(B^T f)^T u \in [-\|B^T f\|_2, \|B^T f\|]$ when $\|u\| \le 1$

$\Downarrow$

$\|B^T f\|_2^2 + f^T d \le g \iff Bu + d \in \mathcal{H} \quad \forall \|u\|_2 \le 1$

(convex in B and d)

$\Downarrow$

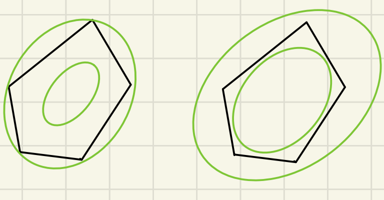
$$\min \ \log\det B$$
$$\text{s.t.} \ \|B^T a_i\|_2 + a_i^T d \le b_i$$

- Efficiency of ellipsoidal approximation

  $C \subseteq \mathbb{R}^n$ convex, bounded, with nonempty interior

  lowner-john ellipsoid, shrunk by a factor n, inside C

  maximum volume inscribed ellipsoid, expanded by a factor n, covers C

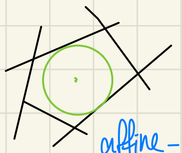


ellipsoids are universal geometric approximators of convex sets

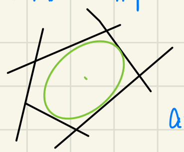
- Centering.

  1. Center of largest inscribed ball ('Chebyshev center') for polyhedron. Can be solved by LP

  2. Center of maximum volume inscribed ellipsoid



affine-variant

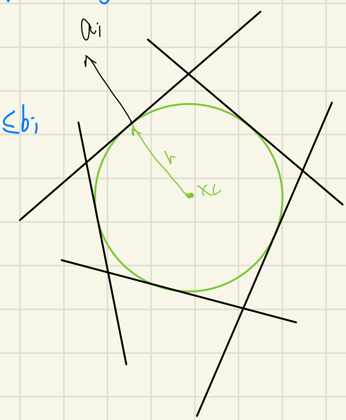


affine-invariant

- Chebyshev center

polyhedron $\mathcal{P} = \{x \mid a_i^T x \le b_i\}$

the inscribed ball parametrized by $\mathcal{B} = \{x_c + u \mid \|u\|_2 \le r\}$



$a_i^T x \le b_i$ for all $x \in \mathcal{B}$ iff

$$\sup_u \{a_i^T(x_c+u) \mid \|u\|_2 \le r\} = a_i^T x_c + r \cdot \|a_i\|_2 \le b_i$$

$$\begin{aligned} &\max \text{ (over } r, x) && r \\ &\text{s.t.} && a_i^T x + r \cdot \|a_i\|_2 \le b_i \end{aligned}$$

- Analytic center

the analytic center of a set of a set of convex inequalities and linear equations $f_i(x) \le 0$ $\quad Fx = g$

$$\begin{aligned} &\min. && -\sum_i \log f_i(x) \\ &\text{s.t.} && Fx = g \end{aligned}$$

eg. analytic center of linear inequalities $a_i^T x \le b$

x_{ac} is the minimizer of

$$\phi(x) = -\sum_i \log(b_i - a_i^T x)$$

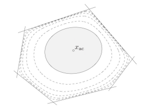
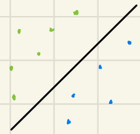


Figure 8.7 The dashed lines show five level curves of the logarithmic barrier function for the inequalities defining the polyhedron C in figure 8.5. The minimizer of the logarithmic barrier function, labeled x_{ac}, is the analytic center of the inequalities. The inner ellipsoid $\mathcal{E}_{inner} = \{x \mid (x - x_{ac})H(x - x_{ac}) \le 1\}$, where H is the Hessian of the logarithmic barrier function at x_{ac}, is shaded.

- Linear discrimination

separate 2 sets of points $\{x_1, \cdots x_N\}$ $\{y_1 \cdots y_M\}$ by a hyperplane

$a^T x_i + b > 0$ $\quad$ $a^T y_i + b < 0$



(homogeneous in a and b
hence equal to)

$\left.\begin{array}{l} a^T x_i + b \geq 1 \quad a^T y_i + b \leq -1 \end{array}\right\}$ LP feasibility problem