$f^*(y) = \sup_x (y^T x - f(x))$ is convex in y

$y^T x - f(x)$ is affine in y $\quad$ $y^T x - f(x)$ is concave in x if f is convex

· Conjugate of Common Convex function

1. Strictly convex quadratic function

$$f(x) = \frac{1}{2} x^T Q x$$

$$f^*(y) = \sup_x y^T x - \frac{1}{2} x^T Q x$$

$$\nabla_x \, y^T x - \frac{1}{2} x^T Q x = y - Qx := 0$$

$$x = Q^{-1} y$$

$$f^*(y) = y^T Q^{-1} y - y^T Q^{-1} Q Q^{-1} y = \frac{1}{2} y^T Q^{-1} y$$

2. Log determinent - inverse

$$f(x) = \log\det(X^{-1})$$

$$f^*(Y) = \sup_X \operatorname{tr}(Y^T X) - \log\det(X^{-1})$$

$$= \sup_X \operatorname{tr}(Y X) + \log\det(X)$$

if $Y \nprec 0$, then Y has an eigenvector v and the associated eigenvalue $\lambda \geq 0$

take $X = I + t v v^T$

$$\operatorname{tr}(YX) + \log\det(X) = \operatorname{tr}(Y) + t \cdot \operatorname{tr}(vv^T Y) + \log\det(I + t v v^T)$$

$$= \operatorname{tr}(Y) + t\lambda + (n-1) + \log(1+t)$$

$$f^*(Y) = +\infty \text{ as } t \to +\infty$$

when $Y \prec 0$

$$\nabla_X \operatorname{tr}(YX) + \log\det X = Y + X^{-1} := 0 \qquad X = -Y^{-1}$$

$$f^*(Y) = \operatorname{tr}(-YY^{-1}) + \log\det(-Y^{-1})$$

$$f^*(Y) = \begin{cases} \log\det(-Y)^{-1} - n & Y \prec 0 \\ +\infty & \text{otherwise} \end{cases}$$

3. log-sum-exp

$$f(x) = \log\left(\sum_i e^{x_i}\right)$$

$$f^*(y) = \sup_x \left(y^T x - \log(1^T e^x)\right)$$

if $y_i < 0$ $\quad f^*(y) \to +\infty$ as $x_i \to +\infty$

if $y \geq 0$ $\quad 1^T y \neq 1$ $\quad$ Choose $x = t \cdot 1$

$$t\, y^T 1 - \log(n \cdot e^t) = t \cdot 1^T y - \log n - t$$

$$f^*(y) = \begin{cases} +\infty & 1^T y > 0 \\ -\infty & 1^T y < 0 \end{cases}$$

when $y \geq 0$ $\quad 1^T y = 0$

$$\nabla_x \, y^T x - \log(1^T e^x) = y - \frac{e^x}{1^T e^x} := 0 \qquad y = \frac{e^x}{1^T e^x}$$

$$f^*(y) = y^T x - \log(\mathbf{1}^T e^x)$$

$$= \sum_i y_i \cdot \log e^{x_i} - \log \sum_j e^{x_j}$$

$$= \sum_i y_i \log e^{x_i} - \sum_i y_i \log \sum_j e^{x_j}$$

$$= \sum_i y_i \log \frac{e^{x_i}}{\sum_j e^{x_j}}$$

$$= \sum_i y_i \log y_i$$

$$f^*(y) = \begin{cases} \sum_i y_i \cdot \log y_i & y \succeq 0 \quad \mathbf{1}^T y = 0 \\ \infty & \text{otherwise} \end{cases}$$

4. norm.

$f(x) = \|x\|$ $\qquad$ $\|x\|_* = \sup_z \{ x^T z \mid \|z\| \le 1 \}$

$f^*(y) = \sup_x \; x^T y - \|x\|$

if $\|y\|_* > 1$ $\quad \|y\|_* = \sup_z \{ y^T z \mid \|z\| \le 1 \} > 1$

take $x = t z^*$

$x^T y - \|x\| = t \cdot (z^{*T} y - \|z^*\|) \to +\infty$ as $t \to \infty$

if $\|y\|_* \le 1$ $\quad \sup_z \{ y^T z \mid \|z\| \le 1 \} \le 1$

$y^T x = y^T \cdot x/\|x\| \cdot \|x\| \le \|x\|$ $\qquad y^T x - \|x\| \le 0$

take $x = 0$ $\quad y^T x = 0$

$$f^*(y) = \begin{cases} 0 & \|y\|_* \le 1 \\ +\infty & \|y\|_* > 1 \end{cases}$$

- Inequality

$f^*(y) = \sup_x \{ x^T y - f(x) \}$ $\qquad f^*(y) \ge x^T y - f(x)$

$f(x) + f^*(y) \ge x^T y$ $\qquad$ Fenchel's Inequality

Young's Inequality for differentiable function