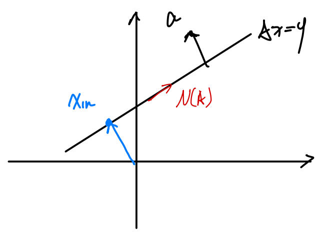
- Least norm solution

$$\min_x \|x\|_2^2 \quad \text{s.t. } Ax=y \qquad x^* = A^T(AA^T)^{-1}y$$

$$\mathcal{L} = \|x\|_2^2 + v^T(Ax-y)$$
$$= \|x\|_2^2 + (A^Tv)^Tx - v^Ty$$
$$\nabla_x \mathcal{L} = 2x + A^Tv := 0$$
$$x^* = -\frac{1}{2}A^Tv$$
$$g(v) = \frac{1}{4}\|A^Tv\|_2^2 - \frac{1}{2}\|A^Tv\|_2^2 - v^Ty$$
$$= -\frac{1}{4}\|A^Tv\|_2^2 - v^Ty \longrightarrow \max_v \ -\frac{1}{4}\|A^Tv\|_2^2 - v^Ty$$
$$\nabla_v g = -\frac{1}{2}AA^Tv - y := 0$$
$$v^* = -2(AA^T)^{-1}y$$
$$x^* = A^T(AA^T)^{-1}y$$

let x_{ln} be the least norm solution.

let $Ax=y$ So $A(x-x_{ln})=0$



$$(x-x_{ln})^Tx_{ln} = (x-x_{ln})^TA^T(AA^T)^{-1}y$$
$$= (A(x-x_{ln}))^T(AA^T)^{-1}y$$
$$= 0$$

$(x-x_{ln}) \perp x_{ln}$

$\|x\|_2^2 = \|x+x_{ln}-x_{ln}\|_2^2 = \|x_{ln}\|_2^2 + \|x-x_{ln}\|_2^2 > \|x_{ln}\|_2^2$ any solution of $Ax=y$ has larger norm than x_{ln}

x_{ln} is the projection of 0 on the solution set $\{x \mid Ax=y\}$

$A^\dagger = A^T(AA^T)^{-1}$ is (a right inverse / the pseudo-inverse) of full row-rank, fat A

$A^\dagger = (A^TA)^{-1}A^T$ is (a left inverse / the pseudo-inverse) of full column-rank, skinny A

$A^T(AA^T)^{-1}A$ gives projection onto $\mathcal{N}(A)^\perp = \mathcal{R}(A^T)$ $\quad$ $A(A^TA)^{-1}A^T$ gives projection on $\mathcal{R}(A)$

$I - A^T(AA^T)^{-1}A$ gives projection on to $\mathcal{N}(A)$

- Least Norm Solution via QR

  project 0 on the solution set $\{x \mid Ax = y\}$

  let $A^T = QR$ $\quad Q^TQ = I$ $\quad R$ upper-triangular and invertible

  $$A^T(AA^T)^{-1}y = QR(R^TQ^TQR)^{-1}y = QR^{-T}y$$

  $$\|x_{ln}\|_2 = \|R^{-T}y\|_2$$

- Least Norm and Regularized Least Square

  $\min \|x\|_2$ $\qquad\qquad$ $\min_x \|Ax-y\|_2^2 + \mu\|x\|_2$

  s.t. $Ax = y$

  $\downarrow$ $\qquad\qquad$ $\downarrow$

  $x_{ln} = A^T(AA^T)^{-1}y$ $\qquad\qquad$ $x(\mu) = (A^TA + \mu I)^{-1}A^Ty$

  as $\mu \to 0$, $(A^TA + \mu I)^{-1}A^T \to A^T(AA^T)^{-1}$ for full row rank, fat A

- Autonomous Linear Dynamical System

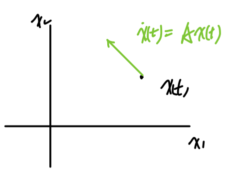
  continuous-time autonomous LDS: $\dot{x} = Ax$

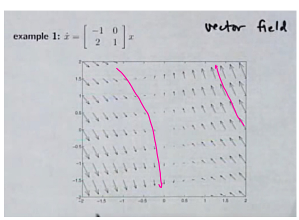
  when A is a scalar $\quad \frac{dx}{dt} = ax$

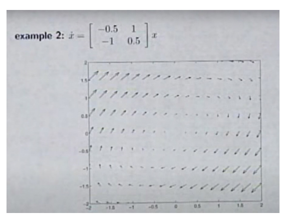
  $$x(t) = e^{at} \cdot x(0)$$

  $x(t)$ is the state

  A is the dynamic matrix (system is time-invariant if A doesn't depend on t)

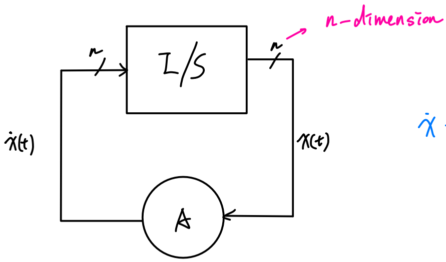




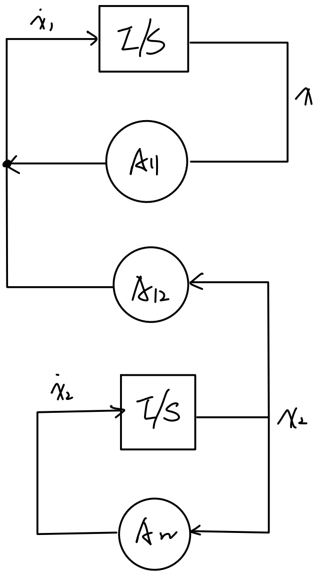


- Block Diagram

block diagram



$\dot{x} = Ax$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$