- Least Square

$$\min_x \|Ax-y\|_2^2$$

$$x^* = (A^TA)^{-1}A^Ty$$

when x square

$$(A^TA)^{-1}A^Ty = A^{-1}A^{-T}A^Ty = A^{-1}y$$

$(A^TA)^{-1}A^T$ is the left inverse. $A^{\dagger}A = (A^TA)^{-1}A^TA = I$

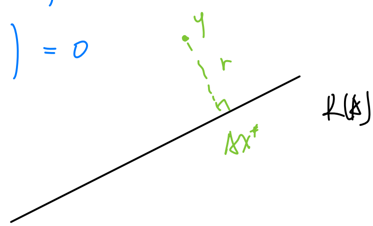
$$Ax^* = A(A^TA)^{-1}A^Ty$$

$A(A^TA)^{-1}A^Ty$ is the projection matrix on $R(A)$

- Orthogonality Principle

$$r = Ax^* - y = A(A^TA)^{-1}A^Ty - y = (A(A^TA)^{-1}A^T - I)y$$

$$r^T(Ax^*) = \left(A(A^TA)^{-1}A^Ty - y\right)^T\left(A(A^TA)^{-1}A^Ty\right) = 0$$



- Least Square with QR

$\min_x \|Ax-y\|_2^2$ $A = QR$ $A \in \mathbb{R}^{m\times n}$ $(m>n)$

$Q \in \mathbb{R}^{m\times n}$ $R \in \mathbb{R}^{n\times n}$ R is upper-triangular. must be invertible

pseudo inverse: $(A^TA)^{-1}A^T = (R^TQ^TQR)^{-1}R^TQ^T = (R^TR)^{-1}R^TQ^T = R^{-1}Q^T$

pseudo inverse: $(A^TA)^{-1}A^T = R^{-1}Q^T$ $x^* = R^{-1}Q^Ty$

$QRx = y$ $x = R^{-1}Q^Ty$ project on the colspace. and cancle the coef

projection matrix: $A(A^TA)^{-1}A^T = AR^{-1}Q^T = QQ^T$

projection matrix: $A(A^TA)^{-1}A^T = QQ^T$

$Ax^* = QQ^Ty$ project on colspace, and reassemble

(note we have $Q^TQ = I$ and $QQ^T \neq I$)

- Least Square with Full QR

$$A = [Q_1 \; Q_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix} \qquad [Q_1 \; Q_2] \in \mathbb{R}^{m\times m}$$

$$\|Ax-y\|_2^2 = \left\| [Q_1 \; Q_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix} x - y \right\|_2^2$$

$\|U x\|_2^2 = x^T U^T U x = \|x\|_2^2$

$$= \left\| [Q_1 \; Q_2]^T [Q_1 \; Q_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix} x - [Q_1 Q_2]^T y \right\|_2^2$$

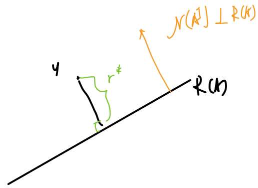
$$= \left\| \begin{bmatrix} R_1 x \\ 0 \end{bmatrix} - [Q_1 Q_2]^T y \right\|_2^2$$

$$= \left\| \begin{bmatrix} R_1 x \\ 0 \end{bmatrix} - \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} y \right\|_2^2$$

$$= \|R_1 x - Q_1^T y\|_2^2 + \|Q_2^T y\|_2^2$$

$$x^* = R_1^{-1} Q_1^T y$$

$$\|r^*\|_2^2 = \|Q_2^T y\|_2^2$$



the optimal residual is y's projection on $\mathcal{N}(A^T)$

$$Ax^* - y = Q_1 R_1 R_1^{-1} Q_1^T y - y$$
$$= Q_1 Q_1^T y - y$$
$$= Q_2 Q_2^T y$$

$Q_1 Q_1^T$ is the projection matrix to $\mathcal{R}(A)$

$Q_2 Q_2^T$ is the projection matrix to $\mathcal{R}(A)^\perp$

$Q = [Q_1 Q_2]$ is square $\quad QQ^T = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}\begin{bmatrix} -Q_1^T- \\ -Q_2^T- \end{bmatrix} = I$

- BLUE property. Best Linear Unbiased Estimator

$y = Ax + v \quad$ A full rank and skinny

a linear estimator $\hat{x} = By = B(Ax+v)$

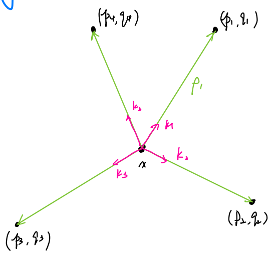
B is unbiased if $\hat{x} = x$ when $v=0$, same as $BA = I$ (B is a left inverse)

Estimation error of unbiased linear estimator is

$x - \hat{x} = x - B(Ax+v) = -Bv \qquad$ we want a "small" left inverse (and $BA = I$)

$\forall$ B s.t. $BA = I$, we have $\|B\|_F^2 \geq \|A^+\|_F^2$

Eg. Navigation



$$\rho_i(x,y) = \sqrt{(x-p_i)^2 + (y-q_i)^2}$$

$$d\rho = A \cdot \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{2(x-p_i)}{\sqrt{(x-p_i)^2+(y-q_i)^2}} & \frac{2(y-q_i)}{\sqrt{(x-p_i)^2+(y-q_i)^2}} \\ — & — \\ — & — \\ — & — \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$= \begin{bmatrix} — k_1^T — \\ — k_2^T — \\ — k_3^T — \\ — k_4^T — \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Eg. Signal Reconstruction

Signal u is piece wise constant, period 1 second. 10 second

$u(t) = x_j$ for $t \in [j, j+1)$ $\quad u(0.5) = x_0 \quad x_j \in \{-1, 0, 1\}$

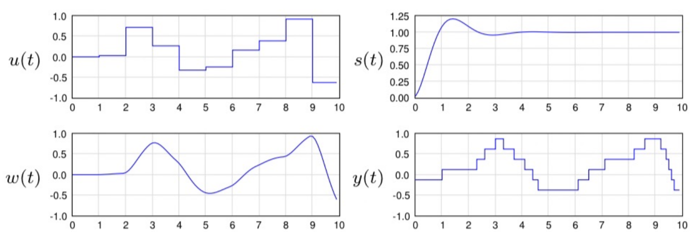
filtered by system with response $h(t)$:

$$w(s) = \int_0^t h(s-\tau)\, u(\tau)\, d\tau$$

Sample $w(s)$ at 10 Hz, we have 100 samples

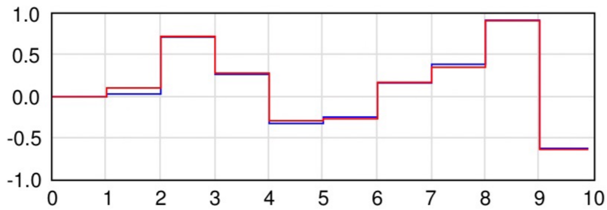
3-bit Quantization:

$y_i = Q(\tilde{y}_i) \qquad Q(a) = \frac{1}{4}\left(\mathrm{round}(4a + \tfrac{1}{2}) - \tfrac{1}{2}\right)$ chose from 8 levels spreads between $[-1, 1]$



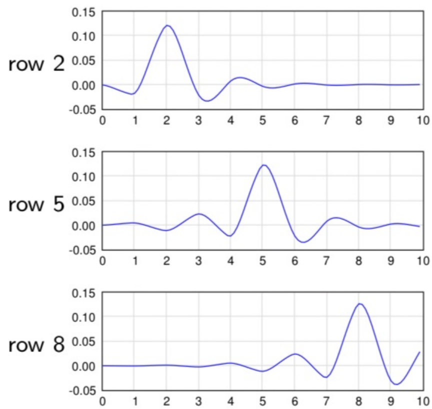
$y = Ax + v \qquad A_{ij} = \int_j^{j+1} h(0.1i - \tau)\, d\tau$

v is the quantization error $|v_i| \le 1/8$



— truth
— LS estimation

Rows of the left-inverse



- some rows of $B_{\text{ls}} = (A^T A)^{-1} A^T$
- rows show how sampled measurements of y are used to form estimate of x_i for $i = 2, 5, 8$
- to estimate x_5, which is the original input signal for $4 \le t < 5$, we mostly use $y(t)$ for $3 \le t \le 7$