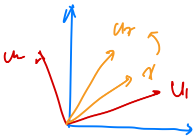
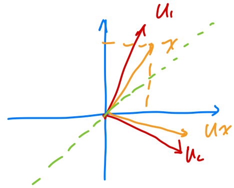
· Interpretation of Ux (U is orthonormal)



rotation



· Gram-Schmidt procedures

given an independent set of vectors $a_1 \cdots a_n$

find the orthonormal basis for these vectors $q_1 \cdots q_n$ $\quad \text{span}(a_1 \cdots a_n) = \text{span}(q_1 \cdots q_n)$

$q_1 = a_1 \qquad q_1 / \|q_1\|_2$

$q_2 = a_2 - (a_2^T q_1) \cdot a_1 \qquad q_2 / \|q_2\|_2$

$q_3 = a_3 - (a_3^T q_1) \cdot a_1 - (a_3^T q_2) \cdot a_2 \qquad q_3 / \|q_3\|$

$\vdots$

if we fail at $q_i / \|q_i\|_2$ by dividing by 0

then a_i can be represented by $a_1 \cdots a_{i-1}$

Can be used to check independence

$\Downarrow$

$$A = QR \qquad \begin{bmatrix} | & | & | & & | \\ q_1 & q_2 & q_3 & \cdots & q_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \vdots & \vdots \\ & r_{22} & \vdots & \vdots \\ & & & \vdots \end{bmatrix} \begin{bmatrix} | & | & & \\ a_1 & a_2 & \cdots & \\ | & | & & \end{bmatrix}$$

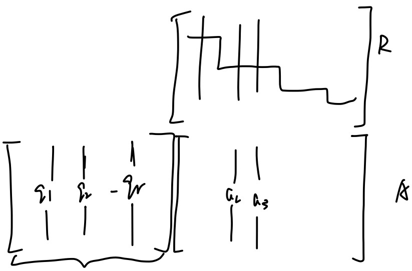
QR factorization

$Q^T Q = I$

$A \in \mathbb{R}^{n \times k} \qquad Q \in \mathbb{R}^{n \times k} \qquad R \in \mathbb{R}^{k \times k}$

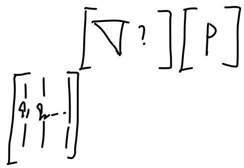
- General Gram-Schmidt

when A is not full col rank $\quad A\in\mathbb{R}^{n\times k} \quad Q\in\mathbb{R}^{m\times n} \quad R=\mathbb{R}^{r\times k}$



$r = \text{rank}(A)$ $\quad$ a_3 is linear comb of a_1 & a_2

With column permutation on R



$= A = Q[\tilde{R}, S]P$

factorization $\quad A=QR \quad Q\in\mathbb{R}^{m\times r} \quad R\in\mathbb{R}^{r\times k}$

eg: check $b\in R(A)$ $\rightarrow$ GS on $[A\ b]$

eg Solve $Ax=b$ $\rightarrow$ GS on $[A, b]$

- Full QR

$$A = [Q_1 \quad Q_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix} [A] \qquad A\in\mathbb{R}^{m\times n}$$

get Q_2: $\quad$ QR on $[A \ I]\in\mathbb{R}^{m\times(n+m)} \quad A\in\mathbb{R}^{m\times n} \quad I\in\mathbb{R}^{m\times m}$

keep track of the num of q produced by A

when we have m q's we stop

$R(Q_1)$ and $R(Q_2)$ are called "complementary subsp"

$\forall\ q\in R(Q_1)\ p\in R(Q_2) \Rightarrow p\perp q \quad (R(Q_1)\perp R(Q_2)) \quad ($

and $R(Q_1)+R(Q_2) = \mathbb{R}^m$

$$A = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} R^T & 0 \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} \Rightarrow \quad \text{null}(A^T) = Q_2$$

Q_2 is an orthonormal basis for $\mathcal{N}(A^T)$

$$\mathcal{R}(A) + \mathcal{N}(A^T) = \mathbb{R}^m$$

$$\mathcal{R}(A)^{\perp} = \mathcal{N}(A^T)$$