- Cayley – Hamilton Theorem

if $p(s) = a_0 + a_1 s + \cdots a_k s^k$ is a polynomial, $A \in \mathbb{R}^{n\times n}$
define $p(A) = a_0 I + a_1 A + \cdots a_k A^k$

$A \in \mathbb{R}^{n\times n}$ is in a $n\times n$-dimension space
$I, A, A^2 \cdots A^{n\times n}$ is linear dependent
$\therefore \exists$ polynomial $p(A) = I + a_1 A + \cdots a_{n\times n} A^{n\times n} = 0$

$A \in \mathbb{R}^{n\times n}$ is in a $n\times n$ dimensional vector space
$\therefore$ I A A^2 $\cdots$ $A^{(n\times n)}$ is linear-dependence
($n\times n+1$ elements in $n\times n$ dimension space)
$\therefore \exists$ polynomial $p(A) = I + a_1 A + \cdots a_{n\times n} A^{(n\times n)} = 0$
However, $\exists$ polynomial with degree n s.t. $p(A) = 0$

actually, $\exists$ a order-n polynomial s.t. $p(A) = I + a_1 A + \cdots a_n A^n = 0$ (A^n is linear combination of $\{I, A, A^2 \cdots A^{n-1}\}$

Cayley – Hamilton Theorem:
for any $A \in \mathbb{R}^{n\times n}$, $\mathcal{X}(A) = 0$, where $\mathcal{X}(s) = \det(sI - A)$

eg. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad \mathcal{X}(s) = \det \begin{bmatrix} 1-s & 2 \\ 3 & 4-s \end{bmatrix} = (s-4)(s-1) - 6 = s^2 - 5s - 2$

$$A^2 - 5A - 2I = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

Corollary: $\forall p \in \mathbb{Z}_+$ $A \in \mathbb{R}^{n\times n}$ $\Rightarrow A^p \in \operatorname{span}\{I, A \cdots A^{n-1}\}$

Proof: $s^p = q(s)\mathcal{X}(s) + r(s)$ $\quad$ compute $\frac{s^p}{\mathcal{X}(s)}$ to get $q(s)$ and $r(s)$

$r(s)$ has degree $< n$
$A^p = q(A)\mathcal{X}(A) + r(A) = r(A)$ $\quad \mathcal{X}(A) = 0$ by Cayley Hamilton theorem
$r(A)$ gives the coef of A^p in terms of $\{I, A \cdots A^{n-1}\}$

eg. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad \mathcal{X}(s) = s^2 - 5s - 2$
let $p = 3$
$s^3 = (s+5)(s^2 - 5s - 2) + (27s + 10)$

$$A^3 = \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix} = 27\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 10 I$$

$$\frac{s^3}{s^2-5s-2} = \frac{s^3 - 5s^2 - 2s + 5s^2 + 2s}{s^2 - 5s - 2}$$
$$= s + \frac{5s^2 + 2s}{s^2 - 5s - 2}$$
$$= s + \frac{5s^2 - 25s - 10 + 27s + 10}{s^2 - 5s - 2}$$
$$= s + 5 + \frac{27s + 10}{s^2 - 5s + 2}$$

$s^3 = (s+5)(s^2 - 5s - 2) + (27s + 10)$
$s^p = q(s)\mathcal{X}(s) + r(s)$

for any $f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_k x^k + \cdots \alpha_n x^n + \cdots$

$$f(A) = \alpha_0 + \alpha_1 A + \cdots + \alpha_k A^k + \cdots \underbrace{\alpha_n A^n + \cdots}_{\text{linear combination of } \{I, A, \cdots A^{n-1}\}}$$

$$= \beta_0 + \beta_1 A + \cdots \beta_{n-1} A^{n-1}$$

for $p = -1$, rewrite the Cayley Hamilton Theorem

$$\mathcal{X}(A) = A^n + a_{n-1} A^{n-1} + \cdots a_0 I = 0$$

$$I = \left(-\frac{a_1}{a_0}\right) A + \left(-\frac{a_2}{a_0}\right) A^2 + \cdots \left(-\frac{1}{a_0}\right) A^n$$

$$A^{-1} = \left(-\frac{a_1}{a_0}\right) I + \left(-\frac{a_2}{a_0}\right) A + \cdots \left(-\frac{a_{n-1}}{a_0}\right) A^{n-2} + \left(-\frac{1}{a_0}\right) A^{n-1}$$

A^{-1} is a linear combination of $\{I, A, \cdots A^{n-1}\}$

$$Ax = y$$

$$x = A^{-1} y = \left(-\frac{a_1}{a_0}\right) y + \left(\frac{-a_2}{a_0}\right) Ay + \cdots \left(-\frac{a_{n-1}}{a_0}\right) A^{n-2} y + \left(-\frac{1}{a_0}\right) A^{n-1} y$$

this will be fast when we have fast simulation $\tilde{y} = Ay$

- Proof Cayley Hamilton Theorem

1° assume A is diagonalizable $A = T\Lambda T^{-1}$

$$\mathcal{X}(s) = (s-\lambda_1)(s-\lambda_2) \cdots (s-\lambda_n)$$

$$\mathcal{X}(A) = \mathcal{X}(T\Lambda T^{-1}) = T\mathcal{X}(\Lambda) T^{-1}$$

$$\mathcal{X}(\Lambda) = (\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I) \cdots (\Lambda - \lambda_n I) = 0$$

$\Lambda - \lambda_i I$ has 0 on the ith diagonal element

2° General case $T^{-1}AT = J$

$$\mathcal{X}(s) = (s-\lambda_1)^{n_1} \cdots (s-\lambda_q)^{n_q}$$

$$\mathcal{X}(A) = \mathcal{X}(TJT^{-1}) = T\mathcal{X}(J) T^{-1}$$

$$\mathcal{X}(J) = \begin{bmatrix} \mathcal{X}(J_1) & & \\ & \ddots & \\ & & \mathcal{X}(J_k) \end{bmatrix}$$

$$\mathcal{X}(J_i) = (J_i - \lambda_1 I)^{n_1} \cdots (J_i - \lambda_i I)^{n_i} \cdots (J_i - \lambda_q I)^{n_q} = 0$$

$$\downarrow$$

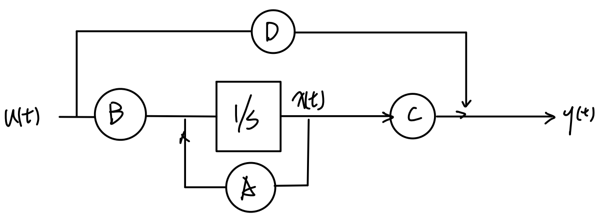
$$\left(\begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix} - \lambda_i I \right)^{n_i} = \left(\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \right)^{n_i} = 0$$

$$\longrightarrow n_i \longleftarrow$$

an up-shift matrix to the n_i power

• Inputs and Outputs

$$\dot{x} = \underbrace{Ax}_{\text{drift term}} + \underbrace{Bu}_{\text{input term}} \qquad y = Cx + Du$$



• Transfer Matrix

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\downarrow \mathcal{L}$$

$$s(\mathcal{L}x)(s) - x(0) = A(\mathcal{L}x)(s) + B(\mathcal{L}u)(s)$$

$$(sI - A)(\mathcal{L}x)(s) = x(0) + B(\mathcal{L}u)(s)$$

$$(\mathcal{L}x)(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}B(\mathcal{L}u)(s)$$

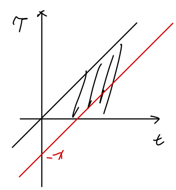
$$\downarrow \mathcal{L}^{-1}$$

$$x(t) = e^{tA}x(0) + \left(e^{tA} * Bu(t)\right)(t)$$

$$= e^{tA}x(0) + \int_0^t e^{(t-T)A}Bu(T)\,dT$$

$$x(t) = e^{tA}x(0) + \int_0^t Bu(T) \cdot e^{(t-T)A}\,dT$$

$$(f * g)(t) = \int_0^t f(T) \cdot g(t-T)\,dT$$



$$\mathcal{L}(f*g)(s) = \int_0^{+\infty} (f*g)(t)\, e^{-st}\,dt$$

$$= \int_{0_t}^{+\infty} \int_{0_T}^{t} f(T)\, g(t-T)\,dT\, e^{-st}\,dt$$

$x = t - T$, $T = t - x$

$$= \int_{0_x}^{+\infty} \int_{0_T}^{+\infty} f(T)\, g(x)\,dT\, e^{-s(T+x)}\,dx$$

$$= \int_{0_x}^{+\infty} \int_{0_T}^{+\infty} f(T)\, e^{-sT}\,dT\; g(x)\, e^{-sx}\,dx$$

$$= (\mathcal{L}f)(s) \cdot (\mathcal{L}g)(s)$$

$$(sI-A)^{-1} = \frac{1}{s}\left(I - \frac{A}{s}\right)^{-1} = \frac{1}{s}\left(I + \frac{A}{s} + \frac{A^2}{s^2} \cdots \right)$$

$$= \left(\frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} \cdots\right)$$

$$\mathcal{L}^{-1}(sI-A)^{-1} = \left(I + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} \cdots\right) = e^{tA}$$

$e^{tA}x(0)$ is the "unforced / autonomous" response

$e^{tA}B$ is "input-to-state" impulse matrix

$(sI-A)^{-1}B$ is "input-to-state" transfer matrix

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= \underbrace{Ce^{tA}x(0)}_{\text{initial condition}} + \underbrace{\int_0^t C\cdot e^{(t-\tau)A}Bu(\tau)\,d\tau}_{\text{propagate } Bu(\tau) \text{ by } (t-\tau),\ \text{and reveal it to output}} + \underbrace{Du(t)}_{\text{final input}}
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{L}y)(s) &= C(\mathcal{L}x)(s) + D(\mathcal{L}u)(t) \\
&= C(sI-A)^{-1}x(0) + C(sI-A)^{-1}B(\mathcal{L}u)(s) + D(\mathcal{L}u)(s) \\
&= C(sI-A)^{-1}x(0) + [C(sI-A)^{-1}B + D](\mathcal{L}u)(s)
\end{aligned}
$$

if no initial condition,

$(\mathcal{L}y)(s) = H(s)\,(\mathcal{L}u)(s)$

$y = h * u$

- Impulse Matrix

  With $x(0)=0$ $\quad (\mathcal{L}y)(s) = [C(sI-A)^{-1}B + D](\mathcal{L}u)(s)$

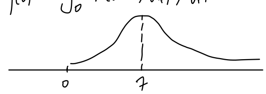
  $y = h * u$

$$y_i = \sum_j \int_0^t h_{ij}(t-\tau)\,u(\tau)\,d\tau$$

(eg y_i is the heigh of the river
u_j is the rainfall on the sector j
for different sector, it takes different time for rain to effect river)

Convolution

$$y(t) = \int_0^t h(t-\tau)\,u(\tau)\,d\tau$$



y depends mostly on the input of 7s ago

eg. u is the rainfall

y is the water in a river

it takes 7 days for water to go into river

- Step Matrix

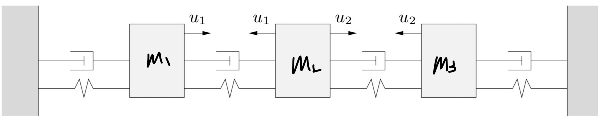
the step response matrix is given by

$$S(t)=\int_0^t h(\tau)\,d\tau$$

$$S_{ij}(t)=\int_0^t h_{ij}(\tau)\,d\tau$$ think $u_j=1$ and $u_i=0$ for $i\neq j$

for Invertible, $S(t)=CA^{-1}(e^{tA}-I)B+D$

eg Spring-mass-dashpot



u_1 is tension between m_1 & m_2

u_2 is tension between m_2 & m_3 (u_1 positive means pushing m_1 m_2 together)

$y\in\mathbb{R}^3$ is displacement of mass 1 2 3

$x=\begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

system is:

$$\dot{x}=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} x+\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

eigenvalues of A are

$$-1.71\pm i0.71,\qquad -1.00\pm i1.00,\qquad -0.29\pm i0.71$$

all decays with oscilation