- eg. Markov chain

$$p(t+1) = P \cdot p(t)$$

$$\begin{bmatrix} p(1|1) & p(1|2) \\ p(2|1) & \end{bmatrix} \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}$$

every column sum is 1 $\quad (\mathbb{1}^T P = \mathbb{1}^T)$

$\mathbb{1}$ is a left eigenvector, with eigenvalue 1

$\det(P-I)=0$, $\therefore$ there's also a right eigenvector with eigenvalue 1

have $Pv = v$, v is the equilibrim distribution / steady state

- Diagonalization

$A = T \Lambda T^{-1}$ $\quad$ cols of T is independent set of eigenvectors

$\Lambda = T^{-1} A T$

A is called defective if not diagonalizable

not all matrices are diagonalizable eg. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

if A has n distinct eigenvalues, then diagonalizable

- Diagonalization and left eigenvector

$A = T\Lambda T^{-1}$ $\quad$ $T^{-1}A = \Lambda T^{-1}$ $\quad$ let $T^{-1} = \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}$ $\quad$ $T = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

$$\begin{bmatrix} - & w_1^T & - \\ - & w_2^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix} A = \begin{bmatrix} - & w_1^T A & - \\ & \vdots & \\ - & w_n^T A & - \end{bmatrix} = \begin{bmatrix} - & \lambda_1 w_1^T & - \\ & \vdots & \\ - & \lambda_n w_n^T & - \end{bmatrix}$$

the row of T^{-1} are left eigenvectors of A, with eigenvalue $\lambda_1 \cdots \lambda_n$

$T^{-1}T = I \Rightarrow w_i^T v_j = \mathbb{1}\{i==j\}$ $\quad$ left & right eigenvectors are dual bases

- Modal Form

$$A = T\Lambda T^{-1}$$

define new coordinate $\tilde{x} = T^{-1}x$

$$T^{-1}AT = \Lambda$$

$$T^{-1}AT\tilde{x} = \Lambda\tilde{x}$$

$$T^{-1}(Ax) = \Lambda\tilde{x} \qquad \widetilde{(Ax)} = \Lambda\tilde{x}$$

the trajectories consist of n independent modes

$$\tilde{x}_i(t) = e^{\lambda_i t}\,\tilde{x}_i(0)$$

when eigenvalues are complex, $\lambda_i = \sigma_i + j\omega_i$, system can be put in real modal form

$$S^{-1}AS = \begin{bmatrix} \boxed{\Lambda_r} & & & \\ & \begin{bmatrix} \sigma_{r+1} & \omega_{r+1} \\ -\omega_{r+1} & \sigma_{r+1} \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} \sigma_n & \omega_n \\ -\omega_n & \sigma_n \end{bmatrix} \end{bmatrix}$$

diagonalization simplifies many expressions

$$(sI - A)^{-1} = (sTT^{-1} - T\Lambda T^{-1})^{-1}$$

$$= \left[T(sI - \Lambda)T^{-1}\right]^{-1}$$

$$= T(sI - \Lambda)^{-1}T^{-1}$$

$$= T\,\mathrm{diag}\left(\frac{1}{s-\lambda_i}\right)T^{-1} = \sum_i \frac{v_i w_i^T}{s-\lambda_i}$$

$$e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} \cdots$$

$$= I + T\Lambda T^{-1} + \frac{T\Lambda^2 T^{-1}}{2!} \cdots$$

$$= T e^{\Lambda} T^{-1}$$

$$= T\,\mathrm{diag}(e^{\lambda_i})T^{-1}$$

$f(A) = T\,\mathrm{diag}(f(\lambda_i))\,T^{-1}$ when f is a polynomial

- Solution via Diagonalization

$\dot{x} = Ax \qquad A = T\Lambda T^{-1}$

$$x(t) = e^{tA} x(0)$$
$$= T e^{t\Lambda} T^{-1} x(0)$$
$$= T \, \mathrm{diag}(e^{t\lambda_i}) T^{-1} x_0$$
$$= \sum_i e^{t\lambda_i} \cdot (v_i w_i^T) x_0$$
$$= \sum_i e^{t\lambda_i} \cdot (w_i^T x_0) v_i$$

$x(0) = TT^{-1} x(0) = \sum_i v_i w_i^T x(0) = \sum_i (w_i^T x(0)) v_i$

any trajectory can be expressed as linear combination of modes

decompose $x(0)$ into modes, propagate each mode, reassemble

eg. divide the eigenvalues into

$R(\lambda_1) \cdots R(\lambda_s) < 0 < R(\lambda_{s+1}) \cdots R(\lambda_n)$

$x(t) = \sum_i e^{\lambda_i t} (w_i^T x(0)) v_i$

$\Rightarrow \sum_i e^{\sigma_i t} \cos(\omega t + \phi) (w_i^T x(0)) v_i$

first s terms exponentially decay
last $n-s$ terms exponentially grow

condition for $x(t) \to 0$: $x(0) \in \mathrm{span}\{v_1 \cdots v_s\}$ stable eigenspace

- Stability of Discrete-Time System

$$A = T\Lambda T^{-1} \qquad x(t+1) = A \cdot x(t)$$

$$x(t) = A^t x(0) = \sum_{i=1}^{n} \lambda_i^t \left(w_i^T x(0)\right) v_i$$

v_i part decays if $|\lambda_i| < 1$ (complex magnitude)
(true even if A not diagonalizable)

$x^+ = Ax$ is stable if all λ_i has magnitude ≤ 1

- Jordan Canonical Form

"as close as a diagonalization when non diagonalizable"

mostly a conceptual tool, rarely used in practice

Any matrix $A \in \mathbb{R}^{n\times n}$ can be written in Jordan canonical form by a similarity transformation

$$T^{-1}AT = J$$

Similarly transformation does not change the determinant $\quad \det AB = \det BA$

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{bmatrix} \qquad \sum_{i=1}^{q} n_i = n$$

$$J_i = \underbrace{\begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}}_{\text{Jordan block with eigenvalue } \lambda_i} \in \mathbb{C}^{n_i \times n_i}$$

J is upper-bidiagonal

J diagonal is the special case of n Jordan blocks of size $n_i = 1 \ \forall i$

Jordan form is unique (up to permutation of blocks)

Can have multiple blocks with same eigenvalue

the characteristic polynomial $\mathcal{X}(s) = \det(sI - A) = (s-\lambda_1)^{n_1} \cdots (s-\lambda_q)^{n_q}$

(distinct eigenvalue $\Rightarrow n_i = 1 \Rightarrow J$ is diagonal)

eg. Jordan blocks with same eigenvalue

$$J = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad J = \left[\begin{array}{ccc|c} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & \\ \hline & & & 1 \end{array}\right]$$

these matrix has eigenvalues $\{1,1,1,1\}$

$$J = \begin{bmatrix} \boxed{\begin{matrix} 1 & 1 \\ & 1 \end{matrix}} & \\ & \boxed{\begin{matrix} 1 & 1 \\ & 1 \end{matrix}} \end{bmatrix}$$

dim Null$(\lambda I - A)$ is the number of Jordan blocks with eigenvalue λ

$$T^{-1}(\lambda_i I - A)T = \lambda_i I - J = \begin{bmatrix} \boxed{\begin{matrix}\lambda_i & 1 \\ & \lambda_i\end{matrix}} & & & \\ & \boxed{\begin{matrix}\lambda_i & 1 \\ & \lambda_i\end{matrix}} & & \\ & & \boxed{} & \\ & & & \ddots \end{bmatrix} - \lambda_i I$$

gives 0 column

eg matrix with eigenvalue $\{-1, -1, -1, 3, 5\}$ may have Jordan form

$$\begin{bmatrix} -1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \boxed{3} & \\ & & & & \boxed{5} \end{bmatrix} \qquad \begin{bmatrix} \boxed{\begin{matrix}-1 & 1 \\ & -1\end{matrix}} & & & \\ & \boxed{-1} & & \\ & & \boxed{3} & \\ & & & \boxed{5} \end{bmatrix}$$

3 and 5 has Jordan Block 1x1
since they both have multiplicity 1

· Generalized Eigenvectors

$$T^{-1}AT = J = \mathrm{diag}(J_1, \dots, J_q)$$

$$T = \begin{bmatrix} | & | & & | \\ T_1 & T_2 & \cdots & T_q \\ | & | & & | \end{bmatrix} \qquad T_i = \begin{bmatrix} | & | & & | \\ v_{i1} & v_{i2} & & v_{in_i} \\ | & | & & | \end{bmatrix}$$

$A v_{i1} = \lambda_i v_{i1}$ $\qquad$ $A v_{ij} = v_{i\,j-1} + \lambda_i v_{ij}$ $\qquad$ $v_{i1} \cdots v_{in_i}$ are generalized eigenvectors

· Jordan Form LDS

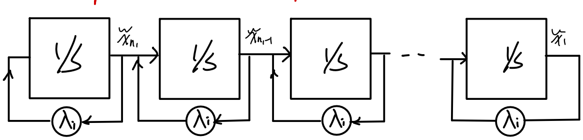
$\dot{x} = Ax \qquad A = TJT^{-1}$

let $\tilde{x} = T^{-1}x$

$\dot{x} = AT\tilde{x} = TJ\tilde{x} \qquad \dot{\tilde{x}} = J\tilde{x}$

$$\begin{bmatrix} \boxed{\begin{matrix}\lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_1\end{matrix}} & & \\ & \boxed{\ddots} & \\ & & \boxed{\ddots} \end{bmatrix} \qquad \tilde{x} = \begin{bmatrix} \begin{bmatrix} \tilde{x}_1 \end{bmatrix} \\ \begin{bmatrix} \tilde{x}_2 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \end{bmatrix} \end{bmatrix}$$

System is decomposed into Independent "Jordan Block System" $\dot{\tilde{x}}_i = J_i \tilde{x}_i$



- Resolvent, Exponential of Jordan Block

$$(sI - J_\lambda)^{-1} = \begin{bmatrix} s-\lambda & -1 & & \\ & s-\lambda & -1 & \\ & & \ddots & -1 \\ & & & s-\lambda \end{bmatrix}^{-1} = \begin{bmatrix} (s-\lambda)^{-1} & (s-\lambda)^{-2} & \cdots & (s-\lambda)^{-k} \\ & (s-\lambda)^{-1} & \cdots & (s-\lambda)^{-k+1} \\ & & & \vdots \\ & & & (s-\lambda)^{-1} \end{bmatrix}$$

$$= (s-\lambda)^{-1} I + (s-\lambda)^{-2} F_1 \cdots (s-\lambda)^{-k} F_{k-1} \qquad F_k = \begin{bmatrix} \overset{\leftarrow k \rightarrow}{\cdots\ 1} & & \\ & \ddots & \\ & & \end{bmatrix}$$

$$e^{tJ_\lambda} = I + tJ_\lambda + \frac{(tJ_\lambda)^2}{2!} \cdots$$

$$= \mathcal{L}^{-1}\left[\frac{I}{s} + \frac{tJ_\lambda}{s^2} + \cdots \frac{(tJ_\lambda)^{n-1}}{s^n} \cdots \right]$$

$$= \mathcal{L}^{-1}\left[(sI - tJ_\lambda)^{-1} \right]$$

$$= \mathcal{L}^{-1}\left[(s-\lambda)^{-1} \right] I + \mathcal{L}^{-1}\left[(s-\lambda)^{-2} \right] F_1 \cdots$$

$$= e^{\lambda t}\left[I + \frac{t}{1!} + \cdots \frac{t^{k-1}}{(k-1)!} F_{k-1} \right]$$

$$= e^{\lambda t} \begin{bmatrix} 1 & t & \cdots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t \cdots & \frac{t^{k-2}}{(k-2)!} \\ & & & \vdots \\ & & & \vdots \end{bmatrix}$$

Jordan blocks correspond to $t^k e^{\lambda t}$ terms

eg

$$\dot{x} = Ax$$

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \qquad x(t) = e^{0t} \begin{bmatrix} 1 & t & \cdots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t \cdots & \frac{t^{k-2}}{(k-2)!} \\ & & & \vdots \\ & & & \vdots \end{bmatrix} x(0)$$

will see ($\lambda = 0$)

$t \cdot e^{\lambda t} = t$

$t^2 e^{\lambda t} = t^2$

$t^3 e^{\lambda t} = t^3$

$$A = \begin{bmatrix} \boxed{0} & & \\ & \boxed{0} & \\ & & \boxed{\begin{matrix} 0 & 1 \\ & 0 \end{matrix}} \end{bmatrix} \qquad x(t) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \boxed{\begin{matrix} 1 & t \\ & 1 \end{matrix}} \end{bmatrix}$$

will see $t \cdot e^{\lambda t}$