- Time Transfer Property

for $\dot{x} = Ax \quad x(t) = e^{tA}x(0)$

the matrix e^{tA} propagate initial condition into $x(t)$: $x(T+t) = e^{tA}x(T)$

eg Sampling a continuous-time system

suppose $\dot{x} = Ax$

sample x at time $t_1 \le t_2 \le \cdots$ let $z(k) = x(t_k)$

then $z(k+1) = e^{(t_{k+1}-t_k)A} z(k)$

eg. Piecewise Constant System

consider time-varying LDS $\dot{x} = A(t)x$

$$A(t) = \begin{cases} A_0 & 0 \le t < t_1 \\ A_1 & t_1 \le t < t_2 \\ \vdots & \end{cases}$$

for $t \in [t_i, t_{i+1}]$

$$x(t) = e^{(t-t_i)A_i} \cdot e^{(t_i - t_{i-1})A_{i-1}} \cdots e^{(t_2-t_1)A_1} \underbrace{e^{t_1 A_0} x(0)}_{x(t_1)}$$

- Qualitative Behavior of $x(t)$

suppose $\dot{x} = Ax \quad x(t) \in \mathbb{R}^n$

then $x(t) = e^{tA}x(0) \quad (\mathcal{L}x)(s) = (sI - A)^{-1}x(0)$

Assume A has distinct eigenvalues $\quad e^{tA} = I + tA + \frac{(tA)^2}{2!} + \cdots \frac{(tA)^n}{n!}$

$x(t) = \sum_{j=1}^{n} \tilde{x}_j(0) \cdot e^{\lambda_j t}$ $\quad \tilde{x}_j(0)$ is the projection of $x(0)$ on jth eigenvector on A

real eigenvalue corresponds to exponentially growing / decaying term

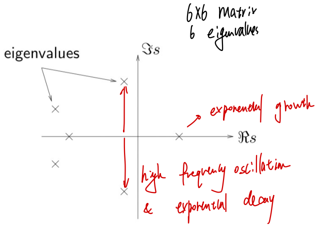
complex eigenvalue $\lambda = \sigma + j\omega$ corresponds to sinusoidal growing/decaying term $e^{\sigma t}[\cos(\omega t) + j\sin(\omega t)]$

vector first-order differential equation gives exponential / sinusoidal growing / decaying in scalar second-order DE $\ddot{x} + a\dot{x} + bx = 0$

$$y = e^{(a+bi)t} = e^{at} e^{ibt} = e^{at}[\cos bt + i \sin bt]$$

if $u(t) + iv(t)$ is a solution to a DE, then $u(t), v(t)$ are solutions

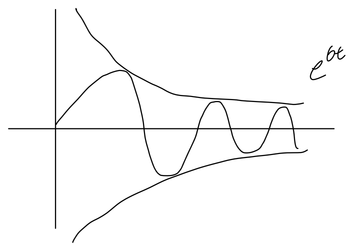
$$(u(t) + iv(t))'' + A(u(t) + iv(t))' + B(u(t) + iv(t)) = 0$$
$$\Downarrow$$
$$[u(t)'' + A u(t)' + B u(t)] + i[v(t)'' + A v(t)' + B v(t)] = 0$$



$$\lambda = \sigma + \omega j$$
$$e^{\lambda t} = e^{\sigma t} e^{t\omega j} = e^{\sigma t}(\cos \omega t + j \sin \omega t) \Rightarrow e^{\sigma} \cos(\omega t + \phi)$$

real(λ) give exponential growth/decay rate

img(λ) gives frequency of oscillation



- Stability

  $\dot{x} = Ax$ is stable if $e^{tA} \to 0$ as $t \to \infty$

  $\to$ $x(t)$ converges to 0 as $t \to +\infty$ regardless of $x(0)$
  all trajectory converges to 0

  $\dot{x} = Ax$ is stable $\Leftrightarrow$ all eigenvalues of A has negative real part

- Eigenvector

  $(\lambda I - A)v = 0$ $\quad$ v is a right eigenvector
  $w^T(\lambda I - A) = 0$ $\quad$ w is a left eigenvector

  even when A is real, eigenvalue and eigenvectors can be complex
  when A and λ are real, can always find real eigenvector

  $$Av = \lambda v$$
  $$A\bar{v} = \overline{Av} = \overline{\lambda v} = \bar{\lambda}\bar{v}$$

  Conjugate symmetry: if A is real, $v \in \mathbb{C}^n$ is a eigenvector with eigen value λ
  then $\bar{v} \in \mathbb{C}^n$ is a eigenvector with eigenvalue $\bar{\lambda}$

Dynamic Interpretation

$\dot{x} = Ax \qquad x(0) = v$

$x(t) = e^{tA} v = \left(I + tA + \frac{(tA)^2}{2!} \cdots \right) v = e^{t\lambda} v$

$x(t) = e^{tA} v$ is the mode of system $\dot{x} = Ax$ (associated with eigenvalue λ)

- Invariant Set

A set $S \subseteq \mathbb{R}^n$ is invariant under $\dot{x} = Ax$ if

$x(t) \in S \Rightarrow x(\tau) \in S \quad \forall \tau \geq t$

once a trajectory enters S, it states in S

eg. for $(\lambda, v) \in \text{eigen}(A)$

then $\{ av \mid a \in R \}$ is an invariant set }

- Complex Eigenvectors

Suppose $Av = \lambda v \quad A \in \mathbb{R}^{n \times n} \quad v \neq 0 \quad \lambda \in \mathbb{C} \quad \lambda = \sigma + j\omega$

let $x = x_r + j x_c$

$Ax = \lambda x = (\sigma + j\omega)(x_r + j x_c) = (\sigma x_r - \omega x_c) + j(\omega x_r + \sigma x_c)$

$$\begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} \begin{bmatrix} x_r \\ x_c \end{bmatrix}$$

the real/img part stays in the span of real and img part

for $a \in \mathbb{C}$, (complex) trajectory $a e^{\lambda t} v$ satisfies $\dot{x} = Ax$, (thus the real part also satisfies)

$x(t) = \text{real}\left(a \cdot e^{\lambda t} v \right)$

$= e^{\sigma t} \, \text{real} \left[(\alpha + j\beta) \cdot (\cos\omega t + j \sin\omega t) (v_r + j v_c) \right] \left[(\alpha \cos\omega t - \beta \sin\omega t) + j(\beta\cos\omega t + \alpha \sin\omega t) \right] \left[v_r + j v_c \right]$

$= e^{\sigma t} \begin{bmatrix} (\alpha\cos\omega t - \beta \sin\omega t) I & -(\beta\cos\omega t + \alpha \sin\omega t) I \end{bmatrix} \begin{bmatrix} v_r \\ v_c \end{bmatrix}$

$v = v_r + j v_c$

$a = \alpha + j\beta$

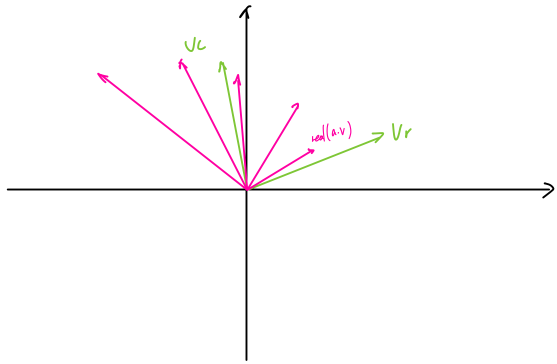
$\lambda = \sigma + j\omega$

$$= \underbrace{e^{\sigma t}}_{\text{growth}} \underbrace{\begin{bmatrix} \alpha I & -\beta I \end{bmatrix}}_{a} \underbrace{\begin{bmatrix} \cos\omega t \, I & -\sin\omega t \, I \\ \sin\omega t \, I & \cos\omega t \, I \end{bmatrix}}_{\text{rotation}} \begin{bmatrix} v_r \\ v_c \end{bmatrix}$$

the real solution stays in the span of real and img part of eigenvector $(\text{span}\{v_r, v_c\})$

$\sigma =$ the real part of λ gives the exponential growth factor

$\omega =$ the img part of λ gives the angular velocity of rotation in plane



· Left Eigenvector

Suppose $w^T A = \lambda w^T \quad w \neq 0$

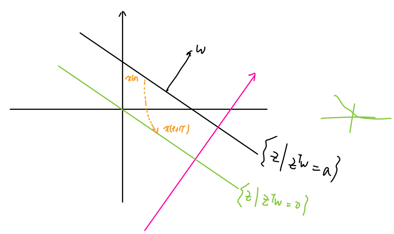
then $\frac{d}{dt}(w^T x) = w^T \dot{x} = w^T A x = \lambda w^T x$

$w^T x$ satisfies the scalar differential equation $\frac{d(w^T x)}{dt} = \lambda (w^T x) \quad w^T x(t) = e^{\lambda t} w^T x(0)$

even if trajectory x is complicated, $w^T x$ is simple

1. if $\lambda \in \mathbb{R}$ $\lambda < 0$, then the halfspace $\{z \mid w^T z \leq a\}$ is invariant $(a \geq 0)$

$w^T x(t) = e^{\lambda t} w^T x(0)$ decreases



2. for $\lambda = \sigma + j\omega$, $\text{real}(w)^T x$ and $\text{img}(w)^T x$ both have the form

$$e^{\sigma t}(\alpha \cos(\omega t) + \beta \sin(\omega t))$$

1. Right eigenvectors are initial conditions from which the resulting motion is simple (i.e. remains in the line (real eigenvalue) or in plane (complex eigenvalue))

2. Left eigenvectors give linear functions of state that are simple, for any initial condition

• Example

$$\dot{x} = \begin{bmatrix} -1 & -10 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x$$

$$(\mathcal{L}f')(s) = \int_0^{+\infty} f'(t)\, e^{-st}\, dt$$

$$= f(t)\, e^{-st}\Big|_0^{+\infty} - \int_0^{+\infty} f(t)\, d e^{-st}$$

$$= -f(0) + s \cdot (\mathcal{L}f)(s)$$

1. Matrix exponential

$$\dot{x} = Ax \qquad \begin{cases} (\mathcal{L}\dot{x})(s) = s \cdot (\mathcal{L}x)(s) - x(0) \\ \mathcal{L}(Ax)(s) = A \cdot (\mathcal{L}x)(s) \end{cases}$$

$$(\mathcal{L}t^n)(s) = \frac{n!}{s^{n+1}}$$

$$s(\mathcal{L}x)(s) - x(0) = A(\mathcal{L}x)(s)$$

$$(\mathcal{L}x)(s) = (sI - A)^{-1} x(0)$$

$$\downarrow$$

$$x(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right](t) \cdot x(0)$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s}(I - A/s)^{-1}\right](t) \cdot x(0)$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s}\left(I + \frac{A}{s} + \frac{A^2}{s^2} + \cdots \frac{A^n}{s^n} \cdots\right)\right](t) \cdot x(0)$$

$$= \mathcal{L}^{-1}\left[\frac{I}{s} + \frac{A}{s^2} \cdots \frac{A^n}{s^{n+1}} \cdots\right](t) \cdot x(0)$$

$$= \left[I + tA + \frac{(tA)^2}{2!} + \cdots \frac{(tA)^n}{n!} \cdots\right] \cdot x(0)$$

$$= \exp(tA) \cdot x(0)$$

$$(I - C)^{-1} = I + C + C^2 + \cdots C^n + \cdots$$

$$(I - C)(I + C + C^2 \cdots C^n + \cdots) = I + C + C^2 \cdots C^n + \cdots$$
$$-(\quad C + C^2 \cdots \quad) = I$$

$$(-1-\lambda)\lambda^2 - 10 - 10\lambda$$

$$= -\lambda^3 - \lambda^2 - 10\lambda - 10$$

2. eigen

$$A - \lambda I = \begin{bmatrix} -1-\lambda & -10 & -10 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (-1-\lambda)\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} + 10\begin{vmatrix} 1 & 0 \\ 0 & -\lambda \end{vmatrix} - 10\begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix}$$

$$= (-1-\lambda)\lambda^2 - 10\lambda - 10$$

$$= -\lambda^3 - \lambda^2 - 10\lambda - 10$$

$$\lambda^3+\lambda^2+10\lambda+10=0$$

$$(\lambda^2+10)(\lambda+1)=0$$

$$\lambda=\pm\sqrt{10}\,j \qquad \lambda=-1$$

3. Stability

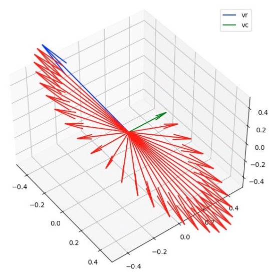
$AV=\lambda V \qquad \lambda=\sigma+jw$

for $a=\alpha+j\beta$, complex trajectory $a\cdot e^{\lambda t}v$ satisfies $\dot{x}=Ax$, (thus the real part also satisfies)

$$x(t)=\text{real}\left(a\cdot e^{\lambda t}v\right)$$

$$=e^{\sigma t}\,\text{real}\left[(\alpha+j\beta)(\cos wt+j\sin wt)(v_r+jv_c)\right]$$

$$=e^{\sigma t}\begin{bmatrix}\alpha I & -\beta I\end{bmatrix}\begin{bmatrix}\cos wt\, I & -\sin wt\, I\\ \sin wt\, I & \cos wt\, I\end{bmatrix}\begin{bmatrix}v_r\\ v_c\end{bmatrix}$$



4. instability

$w^TA=\lambda w^T$

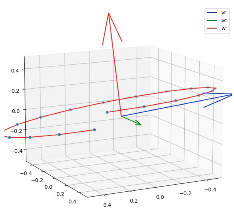
then $\frac{d}{dt}(w^Tx)=w^T\dot{x}=w^TAx=\lambda w^Tx$

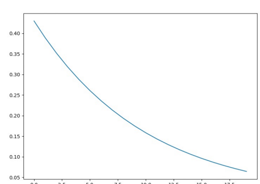
w^Tx satisfies the scalar differential equation $\frac{d\,w^Tx}{dt}=\lambda w^Tx$

$$w^Tx(t)=e^{\lambda t}w^Tx(0)$$

if λ is real

$w^Tx(t)=e^{\lambda t}\,w^Tx(t)$ decrease exponentially





$w^Tx(t)$

if $\lambda=\sigma+jw$

$$\text{Real}\left(w^Tx(t)\right)=\text{Real}\left(e^{(\sigma+jw)t}\,w^Tx(t)\right)$$

$$=e^{\sigma t}\,\text{Real}\left[(\cos wt+j\sin wt)\ \text{Real}(w^Tx(t))+\text{Img}(w^Tx(t))\right]$$