- Laplace transformation

$z : \mathbb{R}_+ \mapsto \mathbb{R}^{p\times q}$

the Laplace Transform $\mathcal{L}z : D \subseteq \mathbb{C} \to \mathbb{C}^{p\times q}$ is

$$(\mathcal{L}z)(s) = \int_0^{+\infty} e^{-st} z(t)\, dt$$

Integral of matrix is done entry-by-entry

D is the domain / region of convergence of z

D includes at least $\{s \mid \Re s > a\}$ where a satisfies $|z_{ij}(t)| \le \alpha e^{at}$ for $t \geq 0$

Derivative property

$$(\mathcal{L}\dot{z})(s) = s\cdot(\mathcal{L}z)(s) - z(0)$$

$$
\begin{aligned}
(\mathcal{L}\dot{z})(s) &= \int_0^{+\infty} e^{-st}\, \dot{z}(t)\, dt \\
&= \int_0^{+\infty} e^{-st}\, dz(t) \\
&= e^{-st}\cdot z(t)\Big|_0^{+\infty} - \int_0^{+\infty} z(t)\, d\, e^{-st} \\
&= 0 - z(0) - \int_0^{+\infty} z(t)\cdot(-s)\, e^{-st}\, dt \\
&= s(\mathcal{L}z)(s) - z(0)
\end{aligned}
$$

- $\mathcal{L}$ Solution of $\dot{x} = Ax$

take Laplace transform of both side

$$s\cdot X(s) - x(0) = A\cdot X(s)$$

$$(sI - A)\, X(s) = x(0)$$

$$X(s) = (sI - A)^{-1} x(0) \qquad (\text{for } sI-A \text{ invertible})$$

take inverse Laplace transform

$$x(t) = \mathcal{L}^{-1}\left[(sI-A)^{-1}x(0)\right] = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]x(0)$$

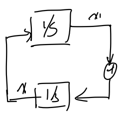
- Resolvent and state transition matrix

$(sI-A)^{-1}$ is the resolvent of A

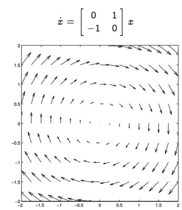
resolvent define for $s \in \mathbb{C}$ except eigenvalue of A

$\Phi(t) = \mathcal{L}^{-1}[(sI-A)^{-1}]$ is the state-transition matrix,
it maps the initial state to state at time t $\quad x(t) = \Phi(t)\,x(0)$

eg. Harmonic Oscillator



Example 1: Harmonic oscillator



$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

$$sI - A = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} / (s^2+1)$$

make sense for all complex number except $\pm j$, which is the eigenvalue of A

$$\Phi(t) = \mathcal{L}^{-1}\left(\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} / (s^2+1) \right) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

a rotation matrix of $-t$ radius

$$x(t) = \Phi(t) \cdot x(0)$$

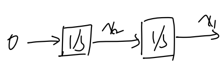
Compare to scalar case: $\dot{x} = ax$

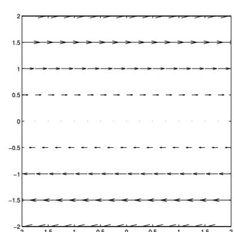
$$x(t) = e^{at} x(0)$$

can only have constant and exponential increase / decrease
but we can have oscillation in vector form

eg. Double Integrator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$





$$sI - A = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}$$

$$\Phi(t) = \mathcal{L}^{-1}\left(\begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix} \right) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0)$$

linear growth in time

- Characteristic Polynomial

$\mathcal{X}_A(s) = \det(sI - A)$ is the characteristic polynomial of A

$\mathcal{X}_A(s)$ is a degree-n polynomial

roots of $\mathcal{X}_A(s)$ are eigenvalues of A

$\mathcal{X}_A(s)$ has real coefficients (when A is real), so eigenvalues are either real or in conjugate pairs

n eigenvalues (may be duplicate)

- Eigenvalues of A and poles of resolvent

i,j entry of the resolvent $(sI-A)^{-1}$ can be expressed via cramer's rule as

$$(-1)^{i+j} \frac{\det \Delta_{ij}}{\det(sI-A)}$$

where Δ_{ij} is $sI-A$ with ith row and jth col removed

i,j entry of the resolvent has the form $\dfrac{\text{degree less than } n \text{ polynomial}}{\text{degree } n \text{ polynomial}}$

- Matrix Exponential

$(I-C)^{-1} = I + C + C^2 + \cdots$ (if series converges) eigenvalues of C ≤ 1

$$\begin{aligned}(I + C + \cdots C^n)(I-C) &= (I + C + \cdots + C^n) \\ &\quad - (C + \cdots + C^n + C^{n+1}) = I - C^{n+1}\end{aligned}$$

$$(sI-A)^{-1} = \frac{1}{s}(I - A/s)^{-1} = \frac{1}{s}\left(I + \frac{A}{s} + \frac{A^2}{s^2} \cdots + \frac{A^n}{s^n} \cdots\right)$$

$$\begin{aligned}\Phi(t) &= \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] \\ &= \mathcal{L}^{-1}\left[\frac{I}{s} + \frac{A}{s^2} + \cdots \frac{A^{n-1}}{s^n} \cdots\right] \\ &= I + tA + \frac{(tA)^2}{2!} \cdots\end{aligned}$$

$$\begin{cases} \Phi(t) = I + tA + \frac{(tA)^2}{2!} + \cdots \\ e^{at} = 1 + ta + \frac{(ta)^2}{2!} + \cdots \end{cases}$$

define the matrix exponential:

$$e^{M} = I + M + \frac{M^2}{2!} + \cdots$$

for $M \in \mathbb{R}^{n\times n}$ (which converges for all M)

with this definition, $\Phi(t) = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = e^{tA}$

$x(t) = e^{tA}x(0)$

Solution of $\dot{x} = Ax$ with $A \in \mathbb{R}^{n\times n}$ constant is $x(t) = e^{tA}x(0)$

(which generalize in scalar case)

$e^{-A} = (e^{A})^{-1}$ $\qquad$ $e^{A+B} = e^{A}e^{B}$ when $AB = BA$

eg. $e^{A+B} \neq e^{A}e^{B}$ in general

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^{A} = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, \qquad e^{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{A+B} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^{A}e^{B} = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}$$

eg. Find Matrix power

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^{tA} = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \qquad e^{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

or $\quad e^{A} = I + A + \frac{A^2}{2!} \cdots = I + A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\qquad$ Since $A^2 = A^3 = \cdots = 0$

· Time Transfer Property

for $\dot{x} = Ax$. $x(t) = \Phi(t)\,x(0) = e^{tA}x(0)$

the matrix e^{tA} is a "time propagator". propagate the initial condition t second forward (backward if $t<0$) in time

$x(t+\tau) = e^{tA}x(\tau) \quad \forall \tau$

$\dot{x} = At$

the forward Euler approximation for small t.

$$x(\tau+t) \approx x(\tau) + t \cdot \dot{x}(\tau) = (I + tA)\,x(\tau)$$

exact solution

$$x(t+\tau) = e^{tA}x(\tau) = \left(I + tA + \frac{(tA)^2}{2!} + \cdots\right)x(\tau)$$