- Autonomous Linear Dynamical System

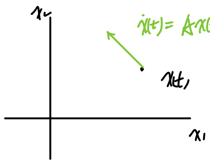
continuous-time autonomous LDS: $\dot{x} = Ax$

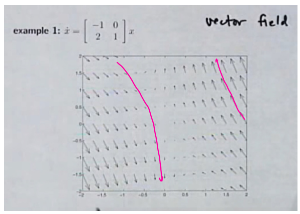
when A is a scalar $\frac{dx}{dt} = ax$

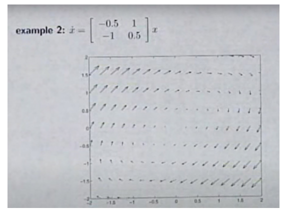
$x(t) = e^{at} \cdot x(0)$

$x(t)$ is the state

A is the dynamic matrix (system is time-invariant if A doesn't depend on t)

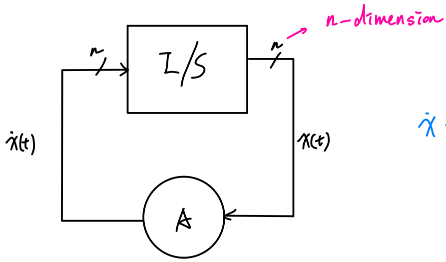




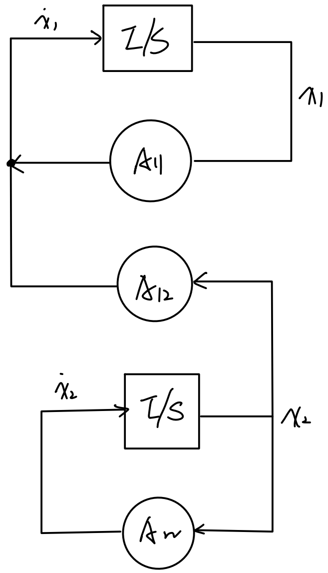


- Block Diagram

block diagram



$$\dot{x} = Ax$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

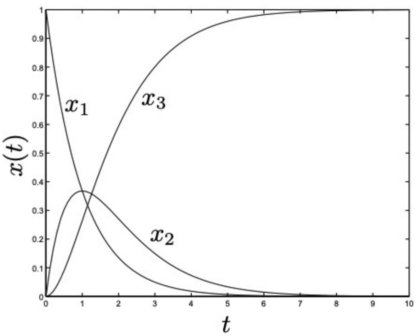
Example: series reaction $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ with linear dynamics

$$\dot{x} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x$$

$$\frac{d}{dt}\left(\mathbb{1}^T x(t)\right) = \mathbb{1}^T \dot{x}(t) = \mathbb{1}^T A \cdot x(t) = 0$$

Time derivative of total amount of material of system is 0

plot for $k_1 = k_2 = 1$, initial $x(0) = (1, 0, 0)$



- Finite-state Discrete-Time Markov chain

  $z(t) \in \{1, \dots n\}$ is a random sequence with

  $$P(z(t+1) = i \mid z(t) = j) = P_{ij}$$

  where $P \in \mathbb{R}^{n \times n}$ is a transition prob matrix

  P is often sparse

$$\begin{bmatrix} P(t_0=0) \\ P(t_0=1) \end{bmatrix}$$

$$\begin{bmatrix} P(t_1=0 \mid t_0=0) & P(t_1=0 \mid t_0=1) \\ P(t_1=1 \mid t_0=0) & \end{bmatrix} \begin{bmatrix} P(t_1=0) \\ \\ \end{bmatrix}$$

- Numerically Integration of Continuous System

  $\dot{x} = Ax \quad x(0) = x_0$

  Forward Euler : $x(t+h) \approx x(t) + h \cdot \dot{x}(t) = x(t) + hAx(t) = (I + hA)x(t)$

  $x(kh) \approx (I + hA)^k x(0)$

  (forward Euler is never used in practice)

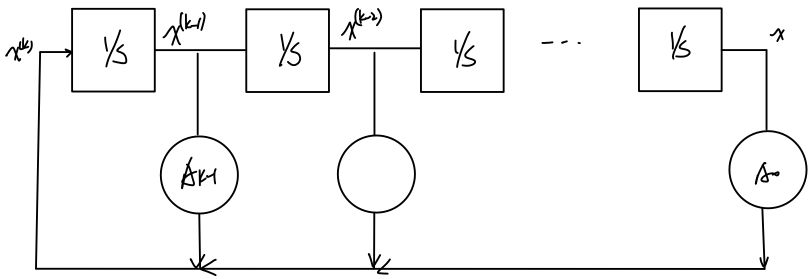
- Higher order linear dynamical system

$$x^{(k)} = A_{k-1} x^{(k-1)} + \cdots A_1 x^{(1)} + A_0 x^{(0)}$$

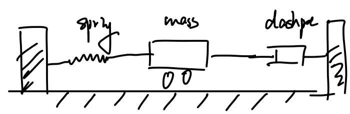
$x^{(k)}$ is the kth derivative of x

$$z = \begin{bmatrix} x \\ x^{(1)} \\ \vdots \\ x^{(k-1)} \end{bmatrix} \quad \dot{z} = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ A_0 & \cdots & & A_{k-1} \end{bmatrix} \begin{bmatrix} x \\ x^{(1)} \\ \vdots \\ x^{(k-1)} \end{bmatrix} = Az$$

upshift and pad with linear combination



eg spring - mass - dashpot



$$M\ddot{q} + D\dot{q} + kq = 0$$

mass dashpot spring

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \quad \dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}k & -M^{-1}D \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

• Linearization Near Equilibrium point

non-linear, time-invariant differential equation

$$\dot{x} = f(x)$$

where $f: \mathbb{R}^n \mapsto \mathbb{R}^n$

suppose x_e is an equilibrium point (i.e. $f(x_e) = 0$)

suppose $x(t)$ near x_e

$$\dot{x}(t) = f(x(t)) \approx f(x_e) + Df(x_e)(x(t) - x_e)$$

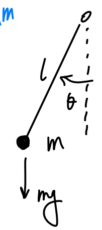
$$= Df(x_e)(x(t) - x_e)$$

$$\dot{x}(t) - \dot{x}_e \approx Df(x_e)(x(t) - x_e)$$

With $(\delta x)(t) = x(t) - x_e$, written

$(\dot{\delta x})(t) \approx Df(x_e)(\delta x)(t)$ like forward euler

e.g. pendulum



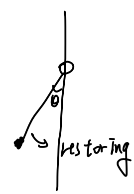
$$ml^2 \ddot{\theta} = -lmg \sin\theta$$

let $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ $\quad \dot{x} = \begin{bmatrix} x_2 \\ -g/l \sin x_1 \end{bmatrix}$

the equilibrium point is $\dot{x} = 0 \Rightarrow \dot{\theta} = 0$ and $\theta = k\pi$

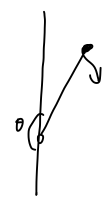
linearize at $x_e = 0$ $\quad (\dot{\delta x}) = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} \delta x$ (pendulum down)

restoring torque



$$\frac{d}{dx_1} -g/l \sin x_1 \Big|_0 = -g/l \cos x_1 \Big|_0 = -g/l$$

linearize at $x_e = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$ $\quad \dot{\delta x} = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix} \delta x$ (pendulum up)



$$\frac{d}{dx_1} -g/l \sin x_1 \Big|_\pi = -g/l \cos x_1 \Big|_\pi = g/l$$

- Linearization Not Always Work

  linearize $\dot{x} = x^3$ near $x_e = 0$

  linearized system has $\dot{\delta x} = 0$

- Linearization Over Trajectory

  suppose $x_{traj} : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfies $\dot{x}_{traj}(t) = f(x_{traj}(t), t)$

  suppose $x(t)$ is another trajectory $\dot{x}(t) = f(x(t), t)$ and is near x_{traj}

$$\frac{d}{dt}(x - x_{traj}) = f(x(t), t) - f(x_{traj}(t), t) \approx D_x f(x_{traj}, t)(x - x_{traj})$$

time-varying dynamical system $\dot{\delta x} = D_x f(x_{traj}, t) \delta x$

eg. Oscillator

$x_{traj}(t)$ is a T-periodic solution for the nonlinear DE

$$\dot{x}_{traj} = f(x_{traj}(t)) \qquad x_{traj}(T+t) = x_{traj}(t)$$

the linearized system is

$$\dot{\delta x} = D_x f(x_{traj}, t) \delta x = A(t) \delta x$$

$A(t)$ is T periodic