- Motivation

Power series $\sum_{n=0}^{\infty} a_n x^n = A(x)$

$$\downarrow$$

$\sum_{n=0}^{\infty} a(n)\, x^n = A(x)$ take in the discrete function $a(n)$ and associate $a(n)$ to $A(x)$

eg. $a(n) = 1$ $A(x) = 1 + x + x^2 + \cdots$

eg. $a(n) = 1/n!$ $A(x) = x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \cdots = e^x$

Continuous Analog

$t \in [0, +\infty)$

$$\int_0^{+\infty} a(t)\, x^t\, dt = A(x)$$

($x < 1$ to converge , $x > 0$ to behave properly)

$$\downarrow \quad x^t = \left(e^{\ln x}\right)^t$$

$x \in (0,1)$ $\ln x \in (-\infty, 0)$ $s = -\ln x = (0, +\infty)$

$$\int_0^{+\infty} f(t)\, e^{-st}\, dt = F(s)$$

Laplace Transform $f(t) \rightsquigarrow F(s)$

$$f(t) \longrightarrow \boxed{\text{Transform}} \longrightarrow F(s) \qquad f(t) \longrightarrow \boxed{\text{Operator}} \longrightarrow g(t)$$

Laplace Transform is linear

$$\mathcal{L}(f+g)(s) = \int_0^{+\infty} \left(f(t) + g(t)\right) \cdot e^{-st}\, dt = (\mathcal{L}f)(s) + (\mathcal{L}g)(s)$$

$$\mathcal{L}(cf)(s) = \int_0^{+\infty} c \cdot f(t)\, e^{-st}\, dt = (c \cdot \mathcal{L}f)(s)$$

Matrix form

let $f : \mathbb{C} \to \mathbb{C}^{p \times q}$

$$(\mathcal{L}f)(s)_{ij} = \int_0^{+\infty} f(t)_{ij} \cdot e^{-st} dt$$

Integral is done entry-by-entry

let $x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

$$\mathcal{L}(Ax(t))_i = \mathcal{L}(a_i^T x(t))(s) = \int_0^{+\infty} a_i^T x(t)\, e^{-st} dt$$

$$= \sum_j a_{ij} \int_0^{+\infty} x(t)_j\, e^{-st} dt$$

$$= a_i^T \mathcal{L}(x(t))(s)$$

$$\mathcal{L}(Ax(t))(s) = A\, \mathcal{L}(x(t))(s)$$

- Laplace Transform

1. $f(t) = 1$ $\qquad 1 \rightsquigarrow \frac{1}{s}$ when $s > 0$

$$(\mathcal{L}f)(s) = \int_0^{+\infty} f(t) \cdot e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{+\infty} = \begin{cases} 1/s & \text{if } s > 0 \\ \text{meaningless} & \text{if } s < 0 \end{cases}$$

2. $e^{at} f(t)$ $\qquad e^{at} f(t) \rightsquigarrow (\mathcal{L}f)(s-a)$

$$\int_0^{+\infty} e^{at} f(t)\, e^{-st} dt = \int_0^{+\infty} f(t)\, e^{-(s-a)t} dt$$

3. $\cos(at)$ $\qquad \cos(at) \to s/(s^2+a^2)$

$$\cos(at) = \frac{e^{iat} + e^{-iat}}{2} \qquad \sin(at) \to a/(s^2+a^2)$$

$$(\mathcal{L}\cos at)(s) = \frac{1}{2}\left[(\mathcal{L}e^{iat})(s) + (\mathcal{L}e^{-iat})(s)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right]$$

$$= s/(s^2+a^2)$$

- Inverse Laplace Transform

$$\frac{1}{S(S+3)} = \frac{1/3}{S} + \frac{-1/3}{S+3}$$

$$\mathcal{L}^{-1}\left(\frac{1}{S(S+3)}\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{S}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{S+3}\right)$$

$$= \frac{1}{3}\cdot 1 - \frac{1}{3}e^{-3t}$$

1. t^n

$$\mathcal{L}(t^n) = \int_0^{+\infty} t^n \cdot e^{-st}\,dt = \int_0^{+\infty} t^n \; d\,\frac{e^{-st}}{-s}$$

$$= t^n \cdot \frac{e^{-st}}{-s}\Big|_0^{+\infty} - \int_0^{+\infty} \frac{e^{-st}}{-s}\,dt^n$$

$$= \lim_{t\to+\infty}\left[\frac{t^n}{\underset{(s>0)}{e^{st}}} \cdot -\frac{1}{s}\right] + \frac{1}{s}\int_0^{+\infty} n\cdot t^{n-1}\,e^{-st}\,dt$$

$$= 0 + \frac{n}{s}\int_0^{+\infty} t^{n-1}\,e^{-st}\,dt$$

$$(\mathcal{L}\,t^n)(s) = \frac{n}{s}\,(\mathcal{L}\,t^{n-1})(s)$$

$$(\mathcal{L}\,t^n)(s) = \frac{n!}{s^n}\,\mathcal{L}(t^0) = \frac{n!}{s^{n+1}}$$

- Existence of $\mathcal{L}$

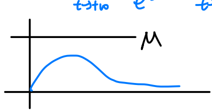
$$(\mathcal{L}f)(s) = \int_0^{+\infty} f(t)\,e^{-st}\,dt$$

f of "exponential type": $\exists\; c, k$ st $|f(t)| \le c\cdot e^{kt}$ $\forall\, t\ge 0$

eg $\sin t$ $\quad |\sin t| \le 1 = 1\cdot e^{0t}$

eg. t^n $\quad |t^n| \le M\cdot e^t$ for some M

$\frac{t^n}{e^t}(0)$ $\quad \lim_{t\to+\infty}\frac{t^n}{e^t} = \lim_{t\to+\infty}\frac{n\cdot t^{n-1}}{e^t} \cdots = \lim_{t\to+\infty}\frac{n!}{e^t} \to 0$



eg. $1/t$

$$\int_0^{+\infty} 1/t \, e^{-st} \, dt$$

when $t \approx 0$ $\quad e^{-st} \approx 1$ $\quad \int_0^{\varepsilon} 1/t \cdot 1 \, dt$ does not converge

$1/t$ is not "exponential type" $\quad$ improper start

eg. e^{t^2}

$\exists t$ s.t $e^{t^2} > e^{kt}$ for any k $\quad$ grow too fast

- Solve ODE with $\mathcal{L}$

$\begin{cases} y'' + Ay' + By = h(t) \\ y(0) = y_0 \quad y'(0) = y_0' \end{cases}$ $\xrightarrow{\mathcal{L}}$ Algebric Equation in $(\mathcal{L}y)(s)$ $\xrightarrow{\text{solve for } (\mathcal{L}y)}$ $(\mathcal{L}y)(s) = \frac{p(s)}{q(s)}$ $\xrightarrow{\mathcal{L}^{-1}}$ $y(t) = (\mathcal{L}^{-1}\mathcal{L}y)(t)$

the Laplace transform must have an initial value problem, assume an init condition if not

- Laplace Transform of Derivative

$$(\mathcal{L}f')(s) = \int_0^{+\infty} f'(t) e^{-st} \, dt$$

$$= e^{-st} f(t) \Big|_0^{+\infty} - \int_0^{+\infty} f(t) \, d e^{-st}$$

$\lim_{t \to +\infty} e^{-st} f(t) = 0$ since $f(t)$ is "exponential type"

$$= -f(0) + s \int_0^{+\infty} f(t) e^{-st} dt$$

$$(\mathcal{L}f')(s) = s(\mathcal{L}f)(s) - f(0)$$

$$(\mathcal{L}f'')(s) = s(\mathcal{L}f')(s) - f'(0)$$

$$= s \cdot \left[s \cdot (\mathcal{L}f)(s) - f(0) \right] - f'(0)$$

$$= s^2 (\mathcal{L}f)(s) - s f(0) - f'(0)$$

$$(\mathcal{L}f'')(s) = s^2 (\mathcal{L}f)(s) - s f(0) - f'(0)$$

- Solve ODE with $\mathcal{L}$

$$\begin{cases} y''-y=e^{-t} \\ y(0)=1 \\ y'(0)=0 \end{cases}$$

$$(\mathcal{L}y'')(s) = s^2(\mathcal{L}y)(s) - sy(0) - y'(0) = s^2(\mathcal{L}y)(s) - s$$

$$\mathcal{L}(e^{-t})(s) = \frac{1}{s+1}$$

$$s^2(\mathcal{L}y)(s) - s - (\mathcal{L}y)(s) = \frac{1}{s+1}$$

$$(\mathcal{L}y)(s) = \frac{s^2+s+1}{(s+1)^2(s-1)} = \frac{-1/2}{(s+1)^2} + \frac{1/4}{(s+1)} + \frac{3/4}{(s-1)}$$

$$\Big\downarrow \mathcal{L}^{-1} \qquad\qquad \Big\downarrow \mathcal{L}^{-1}$$

$$y(t) = -\tfrac{1}{2}\, t\cdot e^{-t} + \tfrac{1}{4} e^{-t} + \frac{3}{4} e^{t}$$

$\dot{x} = Ax$

$$(\mathcal{L}\dot{x})(s) = s\cdot(\mathcal{L}x)(s) - x(0)$$

$$\mathcal{L}(Ax)(s) = A\cdot(\mathcal{L}x)(s)$$

$$s(\mathcal{L}x)(s) - x(0) = A(\mathcal{L}x)(s)$$

$$(sI-A)(\mathcal{L}x)(s) = x(0)$$

$$(\mathcal{L}x)(s) = (sI-A)^{-1}x(0)$$

$$\downarrow$$

$$x(t) = \mathcal{L}^{-1}\left((sI-A)^{-1}\right)x(0)$$

$$(I-C)^{-1} = I + C + C^2 \cdots + C^n$$

$$(I-C)(I+C+\cdots+C^n)$$
$$= (I+C+\cdots C^n) - (C+C^2\cdots - C^{n+1})$$
$$= I - C^{n+1}$$

$$\mathcal{L}^{-1}\left((sI-A)^{-1}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\left(I - A/s\right)^{-1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{I}{s} + \frac{A}{s^2} + \cdots + \frac{A^{n-1}}{s^n} + \cdots\right)$$

$$(\mathcal{L}t^n)(s) = \frac{n!}{s^{n+1}}$$

$$= I + tA + \frac{(tA)^2}{2!} + \cdots$$

$$= e^{tA}$$