EE 261 The Fourier Transform and its Applications
Fall 2007
Problem Set Nine   Due Wednesday, December 5

1. let $f(x,y) = \Pi(x)\Pi(y)$

$(\mathcal{F}f)(\xi_1,\xi_2) = \iint_{\mathbb{R}^2} \Pi(x)\Pi(y) e^{-2\pi i(x\xi_1+y\xi_2)}\,dx\,dy$

$= \int_{-\infty}^{+\infty} \Pi(x) e^{-2\pi i x\xi_1}\,dx \cdot \int_{-\infty}^{+\infty} \Pi(y) e^{-2\pi i y\xi_2}\,dy$

$= (\mathcal{F}\Pi)(\xi_1) \cdot (\mathcal{F}\Pi)(\xi_2)$

$= \operatorname{sinc}(\xi_1) \cdot \operatorname{sinc}(\xi_2)$

$g(x,y) = (f * f)(x,y)$

$(\mathcal{F}g)(\xi_1,\xi_2) = (\mathcal{F}f \cdot \mathcal{F}f)(\xi_1,\xi_2)$

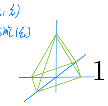
$= \operatorname{sinc}^2(\xi_1) \cdot \operatorname{sinc}^2(\xi_2)$

$g(x,y) = (\mathcal{F}^{-1}\mathcal{F}g)(x,y)$

$= \iint_{\xi_1\,\xi_2} \operatorname{sinc}^2(\xi_1)\operatorname{sinc}^2(\xi_2) e^{2\pi i(\xi_1 x+\xi_2 y)}\,d\xi_1\,d\xi_2$

$= (\mathcal{F}^{-1}\operatorname{sinc}^2)(x) \cdot (\mathcal{F}^{-1}\operatorname{sinc}^2)(y)$

$= \Lambda(x)\Lambda(y)$



1. (10 points) *2D Convolution* Find and sketch the function defined by the following convolution:

$$g(x,y) = \Pi(x)\Pi(y) * \Pi(x)\Pi(y)$$

2. $(\mathcal{F}f)(\xi_1,\xi_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i(x_1\xi_1+x_2\xi_2)} f(x_1,x_2)\,dx_1\,dx_2$

$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}$ $\begin{cases} \xi_1 = \rho\cos\phi \\ \xi_2 = \rho\sin\phi \end{cases}$

$= \int_0^{2\pi}\int_{-\infty}^{+\infty} e^{-2\pi i(r\cos\theta\,\rho\cos\phi + r\sin\theta\,\rho\sin\phi)} \tilde{f}(r)\, r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^{+\infty} e^{-2\pi i r\rho\cos(\theta-\phi)} \tilde{f}(r)\, r\,dr\,d\theta$

$= \int_0^{+\infty}\int_0^{2\pi} e^{-2\pi i r\rho\cos(\theta-\phi)}\,d\theta\ \tilde{f}(r)\, r\,dr$

$= \int_0^{+\infty}\int_0^{2\pi} e^{-2\pi i r\rho\cos\theta}\,d\theta\ \tilde{f}(r)\, r\,dr$

$\frac{1}{2\pi}\int_0^{2\pi} e^{-ia\cos\theta}\,d\theta = J(a)$

$(H\tilde{f})(\rho) = 2\pi\int_0^{+\infty} J(2\pi r\rho)\tilde{f}(r)\, r\,dr$

$(H\tilde{f}(ar))(\rho) = 2\pi\int_0^{+\infty} J(2\pi r\rho)\tilde{f}(ar)\, r\,dr$

$= \frac{1}{a^2} 2\pi \int_0^{+\infty} J(2\pi ar\,\rho/a)\tilde{f}(ar)\, ar\,dar$

$= \frac{1}{a^2} 2\pi \int_{u=ar,\,0}^{+\infty} J(2\pi u\frac{\rho}{a})\tilde{f}(u)\, u\,du$

$= \frac{1}{a^2}(H\tilde{f})(\frac{\rho}{a})$

2. (10 points) *2D Radial Stretch Theorem* If $g(r)$ is a circularly symmetric function with Hankel transform $G(\rho)$, show that the Hankel transform of $g(ar)$ is $\frac{1}{|a|^2}G\left(\frac{\rho}{a}\right)$.

3. (20 points) *2D Fourier Transforms* Find the 2D Fourier Transforms of:

(a). $\sin 2\pi a x_1 = \frac{e^{2\pi i a x_1} - e^{-2\pi i a x_1}}{2i}$

$(\mathcal{F}f)(\xi_1,\xi_2) = \iint_{\mathbb{R}^2} f_1(x_1) f_2(x_2) e^{-2\pi i(x_1\xi_1+x_2\xi_2)}\,dx_1\,dx_2$

$= (\mathcal{F}f_1)(\xi_1)\cdot(\mathcal{F}f_2)(\xi_2)$

$\langle \mathcal{F}e^{2\pi i a x}, \phi\rangle = \langle e^{2\pi i a x}, \mathcal{F}\phi\rangle = \int_{-\infty}^{+\infty} e^{2\pi i a x}(\mathcal{F}\phi)(x)\,dx = (\mathcal{F}^{-1}\mathcal{F}\phi)(a) = \phi(a) = \langle\delta_a,\phi\rangle$

$\left.\begin{array}{l} \mathcal{F}f_1 = \frac{1}{2i}(\delta_a - \delta_{-a}) \\ \mathcal{F}f_2 = \frac{1}{2i}(\delta_b - \delta_{-b}) \end{array}\right\}$ $(\mathcal{F}f)(\xi_1,\xi_2) = \frac{-1}{4}\left(\delta_a(\xi_1) - \delta_{-a}(\xi_1)\right)\left(\delta_b(\xi_2) - \delta_{-b}(\xi_2)\right)$

b). $\mathcal{F}(e^{-\pi r^2}) = e^{-\pi\rho^2}$

$e^{-ar^2} = e^{-\pi(\frac{\sqrt{a}}{\sqrt{\pi}}r)^2}$ $\mathcal{F}\left(e^{-\pi(\frac{\sqrt{a}}{\sqrt{\pi}}r)^2}\right) = \frac{\pi}{a} e^{-\pi(\frac{\sqrt{\pi}}{\sqrt{a}}r)^2} = \frac{\pi}{a} e^{-\frac{\pi^2}{a}r^2}$

c). $f(x,y) = e^{-2\pi i(ax+by)} \cdot \cos(2\pi cx)$

$(\mathcal{F}f)(\xi_1,\xi_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i ax}\cdot\cos(2\pi cx)\, e^{-2\pi i by}\cdot e^{-2\pi i(x\xi_1+y\xi_2)}\,dx\,dy$

$= \int_{-\infty}^{+\infty} e^{-2\pi i ax}\cos(2\pi cx) e^{-2\pi i x\xi_1}\,dx \cdot \int_{-\infty}^{+\infty} e^{-2\pi i by} e^{-2\pi i y\xi_2}\,dy$

$= \mathcal{F}(e^{-2\pi i ax}\cdot\cos(2\pi cx))(\xi_1) \cdot \mathcal{F}(e^{-2\pi i by})(\xi_2)$

$= \left[\mathcal{F}(e^{-2\pi i ax}) * \mathcal{F}(\cos 2\pi cx)\right](\xi_1) \cdot \mathcal{F}(e^{-2\pi i by})(\xi_2)$

$\langle \mathcal{F}e^{-2\pi i ax}, \phi\rangle = \langle e^{-2\pi i ax}, \mathcal{F}\phi\rangle = \int_{-\infty}^{+\infty} e^{-2\pi i ax}(\mathcal{F}\phi)(x)\,dx$

$= (\mathcal{F}^{-1}\mathcal{F}\phi)(-a) = \phi(-a) = \langle\delta_{-a},\phi\rangle$

$= \left[\delta_{-a} * \frac{1}{2}(\delta_c + \delta_{-c})\right](\xi_1)\cdot\delta_{-b}(\xi_2)$

$= \frac{1}{2}\left[\delta_{-a}*\delta_c + \delta_{-a}*\delta_{-c}\right](\xi_1)\cdot\delta_{-b}(\xi_2)$

$= \frac{1}{2}(\delta_{c-a} + \delta_{-c-a})(\xi_1)\cdot\delta_{-b}(\xi_2)$

(a) $\sin 2\pi a x_1 \sin 2\pi b x_2$

(b) e^{-ar^2}

(c) $e^{-2\pi i(ax+by)}\cos(2\pi cx)$

4. $(\mathcal{F}g)(\xi_1,\xi_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(-x_1,x_2) e^{-2\pi i(x_1\xi_1+x_2\xi_2)}\,dx_1\,dx_2$

$= -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(-x_1,x_2) e^{-2\pi i(-x_1\cdot-\xi_1+x_2\xi_2)}\,d{-x_1}\,dx_2$

$= -\int_{+\infty}^{-\infty}\int_{-\infty}^{+\infty} f(u,x_2) e^{-2\pi i(u\cdot-\xi_1+x_2\xi_2)}\,du\,dx_2$

$= (\mathcal{F}f)(-\xi_1,\xi_2)$

$(\mathcal{F}k)(\xi_1,\xi_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_2,x_1) e^{-2\pi i(x_1\xi_1+x_2\xi_2)}\,dx_1\,dx_2$

$= (\mathcal{F}f)(\xi_2,\xi_1)$

$(\mathcal{F}m)(\xi_1,\xi_2) = (\mathcal{F}f)(-\xi_2,-\xi_1)$

(d) $\cos(2\pi(ax+by))$ Hint: Use the addition formula for the cosine.

4. (20 points) Given a function $f(x_1,x_2)$ define

$$g(x_1,x_2) = f(-x_1,x_2), \qquad h(x_1,x_2) = f(x_1,-x_2)$$
$$k(x_1,x_2) = f(x_2,x_1) \qquad m(x_1,x_2) = f(-x_2,-x_1)$$

Find $\mathcal{F}g$, $\mathcal{F}h$, $\mathcal{F}k$, and $\mathcal{F}m$ in terms of $\mathcal{F}f$. Show your work. In each case interpret your result geometrically in terms of an image and its spectrum.

5. (15 points) *2D DFT* Let f be a $M \times N$ matrix (you can think of f as an $M \times N$ image). The 2D DFT of f is given by the following formula:

$$\mathcal{F}f[k,l] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\omega_N^{-ln}\omega_M^{-km}$$

where

$$\omega_N = e^{2\pi i/N}, \quad \omega_M = e^{+2\pi i/M}.$$

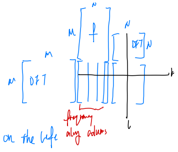
5. a) $(\mathcal{F}f)(k,l) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\, W^{-\frac{mk}{M}} \cdot W^{-\frac{nl}{N}}$

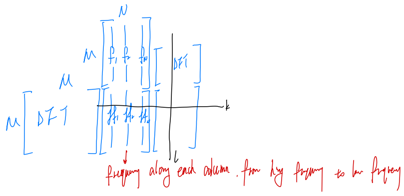
$= \sum_{n=0}^{N-1} W^{-\frac{nl}{N}} \left(\sum_{m=0}^{M-1} f[m,n]\, W^{-\frac{mk}{M}} \right)$

$= \sum_{n=0}^{N-1} W^{-\frac{nl}{N}}\ \mathcal{F}(f[:,n])[k]$

first multiply a M×M DFT matrix on the left

then multiply a N×N DFT matrix on the right





(a) *Separability:* Is the 2D DFT separable? If yes, how can you take advantage of this property to compute the 2D DFT of the matrix f?

b)

$(\mathcal{F}g)[k,l] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g[m,n]\, W^{-\frac{mk}{M}} \cdot W^{-\frac{nl}{N}}$

$= \sum_{m=0}^{N-1}\sum_{n=0}^{M-1} f[m,n] \cdot W^{-\frac{m(k-k_0)}{M}} \cdot W^{-\frac{(l-l_0)}{N}}$

$= (\mathcal{F}f)[k-k_0, l-l_0]$

(b) *Modulation:* Let g be an $M \times N$ image obtained from f in the following way: $g[m,n] = f[m,n]\omega_M^{mk_0}\omega_N^{nl_0}$ where k_0 and l_0 are integers. What is $\mathcal{F}g$ in terms of $\mathcal{F}f$?

(c) *2D convolution* Given 2 matrices f and g of size $M \times N$, their 2D convolution is given by:

$$(f * g)[m,n] = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} f[u,v]g[m-u,n-v]$$

Show that $\mathcal{F}(f * g) = \mathcal{F}f \times \mathcal{F}g$, where on the right we mean the $M \times N$ matrix whose elements are the products of the corresponding elements of $\mathcal{F}f$ and $\mathcal{F}g$.

Hint: Write out the expression for $\mathcal{F}(f * g)$. You'll get a nasty looking expression with four nested sums. But swap the order of summation, change the variable of summation (analogous to changing the variable of integration) and use the fact that the 1D DFT can be computed by summing over any N (or M) consecutive indices.

(c). $(\mathcal{F}f)[k,l] \cdot (\mathcal{F}g)[k,l]$

$= \left(\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n] \cdot W^{-\frac{mk}{M}} \cdot W^{-\frac{nl}{N}} \right) \left(\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g[m,n] \cdot W^{-\frac{mk}{M}} \cdot W^{-\frac{nl}{N}} \right)$

$= \sum_{m_1=0}^{M-1}\sum_{m_2=0}^{M-1}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} f[m_1,n_1]\, g[m_2,n_2]\ W^{-\frac{m_1k}{M}} \cdot W^{-\frac{m_2k}{M}} \cdot W^{-\frac{n_1l}{N}} \cdot W^{-\frac{n_2l}{N}}$

$= \sum_{m_1=0}^{M-1}\sum_{m_2=-m_1}^{M-1-m_1}\sum_{n_1=0}^{N-1}\sum_{n_2=-n_1}^{N-1-n_1} f[m_1,n_1]\, g[m_2,n_2]\ W^{-\frac{m_1k}{M}} W^{-\frac{m_2k}{M}} W^{-\frac{n_1l}{N}} \cdot W^{-\frac{n_2l}{N}}$

$= \sum_{m_1=0}^{M-1}\sum_{n_1=0}^{N-1} f[m_1,n_1] \sum_{m_2=-m_1}^{M-1-m_1}\sum_{n_2=-n_1}^{N-1-n_1} g[m_2,n_2] \cdot W^{-\frac{k(m_1+m_2)}{M}} \cdot W^{-\frac{l(n_1+n_2)}{N}}$

$u = m_2+m_1 \quad v = n_2+n_1$

$= \sum_{m_1=0}^{M-1}\sum_{n_1=0}^{N-1} f[m_1,n_1] \cdot \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} g[u-m_1, v-n_1]\, W^{-\frac{ku}{M}} \cdot W^{-\frac{lv}{N}}$

$= \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} \left(\sum_{m_1=0}^{M-1}\sum_{n_1=0}^{N-1} f[m_1,n_1]\, g[u-m_1, v-n_1] \right) W^{-\frac{ku}{M}} \cdot W^{-\frac{lv}{N}}$

$= \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} (f*g)(u,v)\, W^{-\frac{ku}{M}} \cdot W^{-\frac{lv}{N}}$

$= \mathcal{F}(f*g)[k,l]$

6. (15 points) *Image Segmentation*

Obtain the image dog.jpg from
http://see.stanford.edu/materials/lsoftaee261/dog.jpg

(a) Load the gray-scale image in Matlab using the imread command. Convert the matrix image from type uint8 to double. Plot the initial image.

(b) Now, consider the matrices:

$$\mathbf{B_x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B_y} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the Matlab conv2 command (with the option 'same' - run help for details), filter the initial image by convolving it with the matrix $\mathbf{B_x}$. Filter the initial image once more by convolving it with the matrix $\mathbf{B_y}$. Find all the entries of the two filtered images that are greater than 10 and make the corresponding entries of the initial image equal to 255. Plot the resulting image. What do you observe?

(c) Repeat the steps in part b) only this time instead of filtering the initial image, use a 'smoothed' version of it. Generate the smoothed version by passing the initial image through an ideal Low Pass 2-D filter. Now, increase / decrease the LP filter's width. In each case, plot the 'smoothed image' and the final resulting image. What do you observe?

For filter: http://see.stanford.edu/materials/lsoftaee261/LP_filter.txt