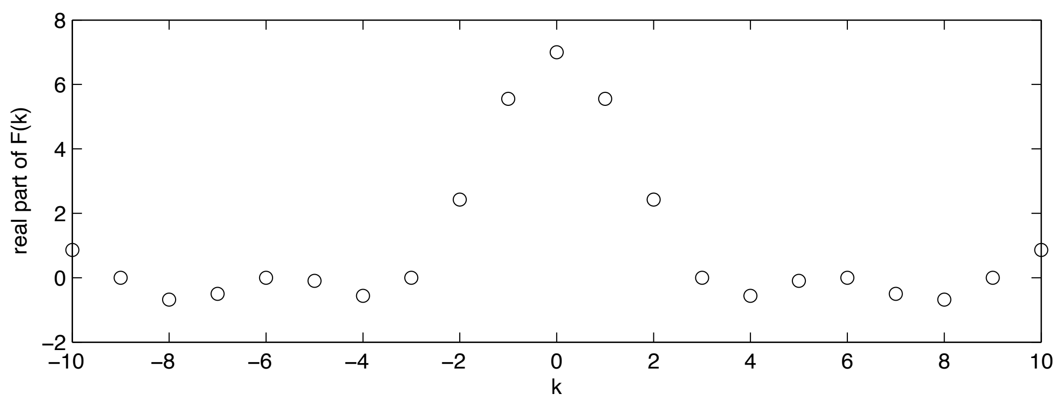
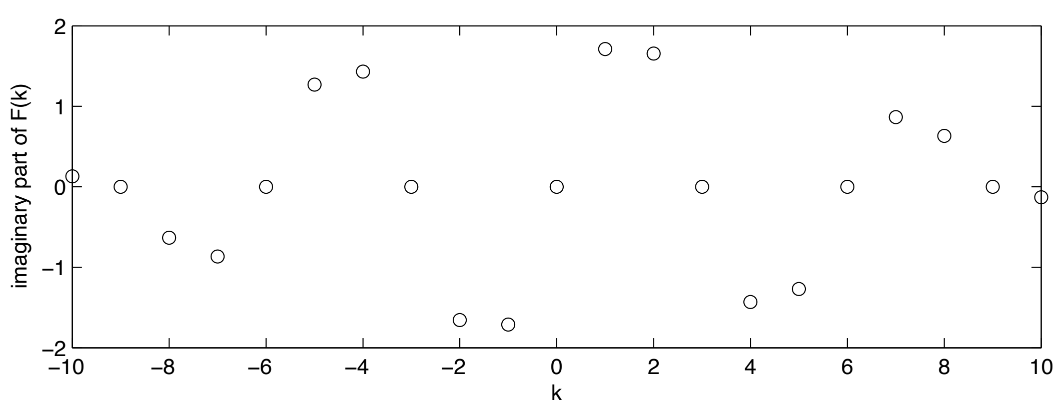
EE 261 The Fourier Transform and its Applications
Fall 2007
Problem Set Eight   Due Wednesday, November 28

1. (20 points) *A True Story*: Professor Osgood and a graduate student were working on a discrete form of the sampling theorem. This included looking at the DFT of the discrete rect function

$$\underline{f}[n] = \begin{cases} 1, & |n| \le \frac{N}{4} \\ 0, & -\frac{N}{2}+1 \le n < -\frac{N}{4}, \quad \frac{N}{4} < n \le \frac{N}{2} \end{cases}$$

The grad student, ever eager, said 'Let me work this out.' A short time later the student came back saying 'I took a particular value of N and I plotted the DFT using MATLAB (their FFT routine). Here are plots of the real part and the imaginary part.'





   (a) Produce these figures.

Professor Osgood said, 'That can't be correct.'

(b) Is Professor Osgood right to object? If so, what is the basis of his objection, and produce the correct plot. If not, explain why the student is correct.

2. (20 points) *Linearity and time-invariance.* State whether the following systems are linear or non-linear, time-invariant or time-variant, and why. Assume that $v(t)$ is the input and $w(t)$ is the output for all systems. *No credit will be given for answers without explanations or proofs!*

   (a) $w(t) = v(t)\cos(\omega t)$

   (b) $w(t) = \sin(v(t))$

   (c) $w(t) = \int_{-\infty}^{\infty} v(\tau)e^{-2\pi i t\tau}\, d\tau$

   (d) $w(t) = \frac{d}{dt}v(t)$

   (e) $w(t) = \cos(\omega t + v(t))$

(a) of course linear
$v(t-a)\cos(wt) \neq w(t-a)$ ∴ time-variant
(b) of course non-linear
$\sin(v(t-a)) = w(t-a)$ ∴ time-invariant
(c) $w(t) = (\mathcal{F}v)(t)$
$(\mathcal{F}v_1)(t) + (\mathcal{F}v_2)(t) = (\mathcal{F}(v_1+v_2))(t)$
∴ linear
$\int_{-\infty}^{+\infty} v(s-a)\cdot e^{-2\pi i s t}\, ds$
$= \int_{-\infty}^{+\infty} v(u)\, e^{-2\pi i (u+a) t}\, du$ ($u = s-a$)
$\neq w(t-a)$ ∴ time-variant
(d). $w(t) = \frac{d}{dt}v(t)$ of course linear
$\frac{d}{dt}\tau_a v(t) = \frac{d}{dt}v(t-a) = w(t-a)$
∴ time-invariant
(e). $w(t) = \cos(wt + v(t))$
non-linear
time-variant

3. (15 points) *Zero-order hold.* The music on your CD has been sampled at the rate 44.1 kHz. This sampling rate comes from the Sampling Theorem together with experimental observations that your ear cannot respond to sounds with frequencies above about 20 kHz. (The precise value 44.1 kHz comes from the technical specs of the earlier audio tape machines that were used when CDs were first geting started.)

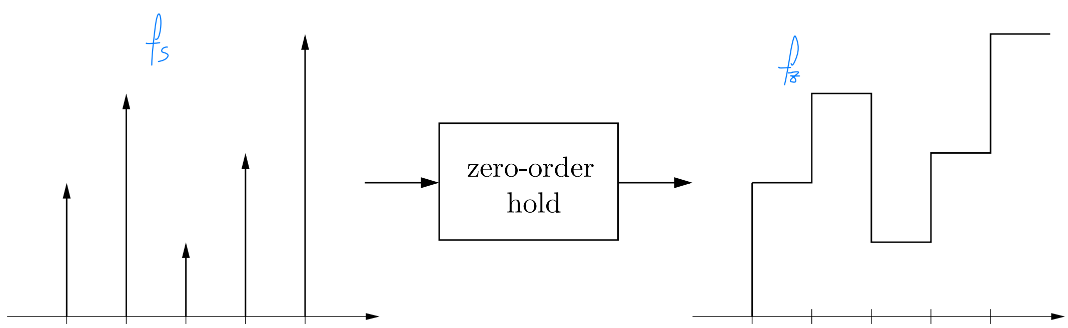
   A problem with reconstructing the original music from samples is that interpolation based on the sinc function is not physically realizable – for one thing, the sinc function is not time-limited. Cheap CD players use what is known as 'zero-order hold'. This means that the value of a given sample is *held* until the next sample is read, at which point that sample value is held, and so on.

   Suppose the input is represented by a train of δ-functions, spaced $T = 1/44.1$ msec apart with strengths determined by the sampled values of the music, and the output looks like a staircase function. The system for carrying out zero-order hold then looks like the diagram, below. (The scales on the axes are the same for both the input and the output.)

(a). $f_z(t) = \sum_{n=-\infty}^{+\infty} f_s(nT)\cdot \tau_{nT+\frac{T}{2}}\Pi_T(t)$
of course linear
when input f is shifted
$\sum_{n=-\infty}^{+\infty} f(nT-mT)\cdot \Pi_T(t-nT-\frac{T}{2}-mT+mT)$
$= \sum_{n=-\infty}^{+\infty} f((n-m)T)\cdot \tau_{(n-m)T+\frac{T}{2}}\Pi_T(t-mT)$
$= \sum_{u=-\infty}^{+\infty} f(uT)\, \tau_{uT+\frac{T}{2}}\Pi_T(t-mT)$
$= f_z(t-mT) = \tau_{mT} f_z(t)$
∴ is time-invariant

(2). $Lv(x) = \int_{-\infty}^{+\infty} h(x,y)\, v(y)\, dy$
when y is a sample point and $x < y + \frac{T}{2}$
v has effect on Lv
∴ $h(x,y) = \begin{cases} \Pi_T(x-y-\frac{T}{2}) & \text{if } y = mT \\ 0 & \text{else} \end{cases}$
$L\delta_y = \tau_{y+\frac{T}{2}}\Pi_T(x)$

(3). the transfer function is $\mathcal{F}\tau_{\frac{T}{2}}\Pi_T(x)$
$(\mathcal{F}\tau_{\frac{T}{2}}\Pi_T)(s) = e^{-2\pi i s\frac{T}{2}}\cdot(\mathcal{F}\Pi(t/T))(s)$
$= T\cdot e^{-\pi i s T}\cdot \mathrm{sinc}(Ts)$



   (a) Is this a linear system? Is it time invariant for shifts of integer multiples of the sampling period?

   (b) Find the impulse response for this system.

   (c) Find the transfer function.

4. (20 points) Let $w(t) = Lv(t)$ be an LTI system with impulse response $h(t)$; thus $w(t) = (h * v)(t)$.

$(Lv)(x) = \int_{-\infty}^{+\infty} (L\delta_y)(x)\, v(y)\, dy = \int_{-\infty}^{+\infty} (L\delta)(x-y)\, v(y)\, dy = \int_{-\infty}^{+\infty} h(x-y)\, v(y)\, dy = (h * v)(x)$

(b). $w(x) = \int_{-\infty}^{+\infty} h(y-x)\, v(y)\, dy$

$= \int_{-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} h(k)\,\mathrm{sinc}(y-x-k)\right)\left(\sum_{l=-\infty}^{+\infty} v(l)\,\mathrm{sinc}(y-l)\right) dy$

$= \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} h(k)\,v(l) \int_{-\infty}^{+\infty} (\tau_k\,\mathrm{sinc})(y-x)\,(\tau_l\,\mathrm{sinc})(y)\, dy$

$= \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} h(k)\,v(l)\, (\tau_k\,\mathrm{sinc} * \tau_l\,\mathrm{sinc})(x)$

$(\tau_k\,\mathrm{sinc} * \tau_k\,\mathrm{sinc})(x) = [\mathcal{F}^{-1}\mathcal{F}(\tau_k\,\mathrm{sinc} * \tau_l\,\mathrm{sinc})](x)$

$= [\mathcal{F}^{-1}(\mathcal{F}\tau_k\,\mathrm{sinc} \cdot \mathcal{F}\tau_l\,\mathrm{sinc})](x)$

$= \mathcal{F}^{-1}(e^{-2\pi i s(k+l)} \cdot \Pi(s))(x)$

$= (\mathcal{F}^{-1}e^{-2\pi i s(k+l)} * \mathcal{F}^{-1}\Pi(s))(x)$

$= [(\mathcal{F}e^{-2\pi i s(k+l)})^{-} * \mathrm{sinc}](x)$

$\langle \mathcal{F}e^{-2\pi i s(k+l)}, \phi\rangle = \langle e^{-2\pi i s(k+l)}, \mathcal{F}\phi\rangle$

$= \int_{-\infty}^{+\infty} e^{-2\pi i s(k+l)}\,\mathcal{F}\phi(s)\, ds$

$= \mathcal{F}^{-1}\mathcal{F}\phi(-k-l) = \langle \delta_{-k-l}, \phi\rangle$

$[\delta_{k+l} * \mathrm{sinc}](x) = (\tau_{k+l}\,\mathrm{sinc})(x) = \mathrm{sinc}(x-k-l)$

$\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} h(k)\,v(l)\,\mathrm{sinc}(x-k-l)$

(a). $w = h * v$

$\mathcal{F}w = \mathcal{F}h \cdot \mathcal{F}v$

if $\mathcal{F}h$ is bandlimited to $[-B, B]$

$(\mathcal{F}w)(s) = 0$ for $s < -B$ or $s > B$

$\therefore$ w is bandlimited

(a) If $h(t)$ is bandlimited show that $w(t) = Lv(t)$ is bandlimited regardless of whether $v(t)$ is bandlimited.

(b) Suppose $v(t)$, $h(t)$, and $w(t) = Lv(t)$ are all bandlimited with spectrum contained in $-1/2 < s < 1/2$. According to the sampling theorem we can write

$$v(t) = \sum_{k=-\infty}^{\infty} v(k)\,\mathrm{sinc}(t-k) \quad h(t) = \sum_{k=-\infty}^{\infty} h(k)\,\mathrm{sinc}(t-k) \quad w(t) = \sum_{k=-\infty}^{\infty} w(k)\,\mathrm{sinc}(t-k)$$

Express the sample values $w(k)$ in terms of the sample values of $v(t)$ and $h(t)$. (You will need to consider the convolution of shifted sincs.)

5. (20 points) Let $a > 0$ and define the (continuous) running average of a signal $f(t)$ to be

$$Lf(t) = \frac{1}{2a}\int_{t-a}^{t+a} f(x)\, dx$$

(a). obviously linear

$\frac{1}{2a}\int_{t-a}^{t+a} \tau_b f(x)\,dx = \frac{1}{2a}\int_{t-a}^{t+a} f(x-b)\, d(x-b)$

$= \frac{1}{2a}\int_{t-a-b}^{t+a-b} f(u)\, du = (Lf)(t-b) = (\tau_b Lf)(t)$

$\therefore$ time-invariant

$(Lf)(x) = \frac{1}{2a}\int_{x-a}^{x+a} f(y)\, dy$

$= \int_{-\infty}^{+\infty} (\tau_x \frac{1}{2a}\Pi_{2a})(y)\, f(y)\, dy$

$= (\frac{1}{2a}\Pi_{2a} * f)(x)$

$\therefore$ the impulse-response is $h(x,y) = (\tau_x \frac{1}{2a}\Pi_{2a})(y)$

the transfer function is $(\mathcal{F}\frac{1}{2a}\Pi_{2a})(s) = \mathrm{sinc}(2as)$

(c). $Lf = \frac{1}{2a}\Pi_{2a} * f$

$LLf = \frac{1}{2a}\Pi_{2a} * (Lf)$

$= \frac{1}{4a^2}\Pi_{2a} * \Pi_{2a} * f$

$= (\frac{1}{2a} \cdot \Lambda(\frac{t}{2a}) * f)$

$\therefore$ the impulse-response is $h(x,y) = \frac{1}{2a}\tau_y \Lambda(\frac{x}{2a})$

the transfer function is $[\mathcal{F}\frac{1}{2a}\Lambda(\frac{t}{2a})](s) = \mathrm{sinc}^2(2as)$

(a) Show that L is an LTI system and find its impulse response and transfer function.

(b) What happens to the system as $a \to 0$? Justify your answer based on properties of the impulse response.

(c) Find the impulse response and the transfer function for the cascaded system $M = L(Lf)$. Simplify your answer as much as possible, *i.e.*, don't simply express the impulse response as a convolution.

(d) Three signals are plotted (approximately) below. One is an input signal $f(t)$. One is $Lf(t)$ and one is $Mf(t)$. Which is which, and why?

(d). $Lf = \frac{1}{2a}\Pi_{2a} * f$

$(\mathcal{F}Lf)(s) = \mathrm{sinc}(2as) \cdot \mathcal{F}f(s)$

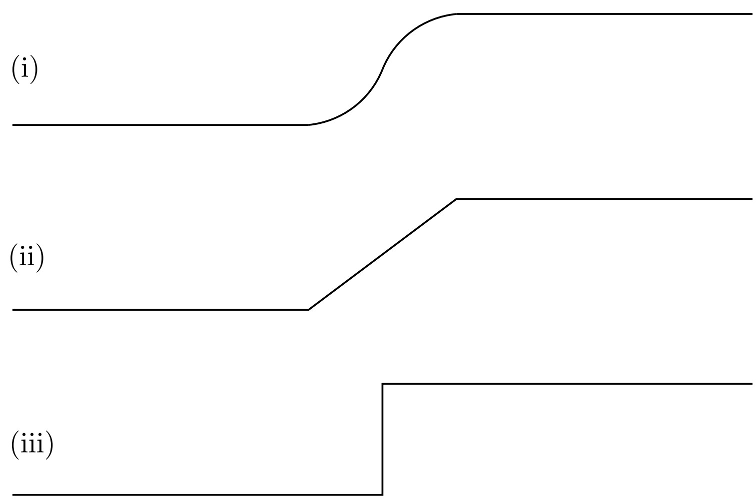
L lower the amplitude of high freq of f

thus smoothing f

$\therefore$ (iii) is f

(ii) is Lf

(i) is $Mf = LLf$



6. (15 points) *Upsampling and Downsampling discrete signals:* We are given a discrete-time signal x_n, which was obtained by sampling a continuous-time $x(t)$ at frequency much higher than the Nyquist rate. Let $x_n = x(nT_s)$. Sometimes for an application, we need the samples of the signal $x(t)$ at a different sampling rate, say αT_s, where $\alpha > 0$. If $\alpha > 1$, it is downsampling else it is upsampling. One way to obtain the sample points at a different sampling frequency, is to reconstruct the signal $x(t)$ (which can be done as the original samples were obtained by sampling above the Nyquist rate) from the samples x_n and then resample it at the new frequency. However, due to non-idealities of the D/A and the A/D converters, this approach is not desirable. In this question, we want to obtain the new sample points directly by using x_n.

(a) Find the expression for $w_n = x\left(n\left(MT_s\right)\right)$ in terms of x_n, where M is an integer.

(b) Find the expression for $y_n = x\left(n\left(\frac{T_s}{M}\right)\right)$ in terms of x_n, where M is an integer. If the resulting expression looks complicated, it will motivate the use of linear interpolation or nearest-neighbor interpolation as used in HW-5.

(c) Find the expression for $z_n = x\left(n\left(\alpha T_s\right)\right)$ in terms of x_n, where $\alpha > 0$ is a rational number $\alpha = \frac{P}{Q}$, where P and Q are integers. Give some insight into how would you accomplish a change in sampling frequency for any $\alpha > 0$ (real number).

(a). down sampling

$$W_n = X(nM\,T_s) = X_{Mn}$$

(b). up sampling

$$y_n = X\left(n \cdot \frac{T_s}{M}\right) = X\left(\frac{n}{M}T_s\right)$$

$$= \sum_{k=-\infty}^{+\infty} X(kT_s)\,\mathrm{sinc}\left(\frac{1}{T_s}\left(\frac{n}{M}T_s - kT_s\right)\right)$$

$$= \sum_{k=-\infty}^{+\infty} X_k\,\mathrm{sinc}\left(\frac{n}{M} - k\right)$$

(C). $Z_n = X(n \cdot \alpha T_s) = X\left(n \cdot \frac{P}{Q} T_s\right)$

first perform upsampling:

$$y_n = X\left(n \cdot \frac{T_s}{Q}\right) = \sum_{k=-\infty}^{+\infty} X_k\,\mathrm{sinc}\left(\frac{n}{Q} - k\right)$$

then perform downsampling

$$Z_n = X\left(n \cdot P \cdot \frac{T_s}{Q}\right) = y_{Pn} = \sum_{k=-\infty}^{+\infty} X_k\,\mathrm{sinc}\left(\frac{Pn}{Q} - k\right)$$