EE 261 The Fourier Transform and its Applications
Fall 2007
Problem Set Two   Due Wednesday, October 10

1. (25 points) *A periodic, quadratic function and some surprising applications*

   Let $f(t)$ be a function of period $T = 2$ with

$$f(t) = t^2 \quad if \quad 0 \le t < 2.$$

   Here's the picture.

a) $\hat{f}(k) = \frac{1}{2}\int_0^2 t^2 e^{-\pi ikt}\, dt$

$= \frac{1}{2} \frac{1}{-\pi ik} \int_0^2 t^2\, d e^{-\pi ikt}$

$= \frac{1}{2} \frac{1}{-\pi ik} \left[t^2 \cdot e^{-\pi ikt}\Big|_0^2 - \int_0^2 e^{-\pi ikt}\, dt^2 \right]$

$= \frac{1}{2} \frac{1}{-\pi ik} \left[4 \cdot e^{-2\pi ik} - 2\int_0^2 t \cdot e^{-\pi ikt}\, dt \right]$

$= \frac{2}{-\pi ik} + \frac{1}{\pi ik} \int_0^2 t \cdot e^{-\pi ikt}\, dt$

$= \frac{2}{-\pi ik} - \frac{1}{(\pi ik)^2} \int_0^2 t \cdot d e^{-\pi ikt}$

$= \frac{2}{-\pi ik} - \frac{1}{(\pi ik)^2} \left[t \cdot e^{-\pi ikt}\Big|_0^2 - \int_0^2 e^{-\pi ikt}\, dt \right]$

$= \frac{2}{-\pi ik} - \frac{1}{(\pi ik)^2} \left[2 - \left(\frac{1}{-\pi ik}\right) e^{-\pi ikt}\Big|_0^2 \right]$

$= \frac{2}{-\pi ik} - \frac{2}{(\pi ik)^2}$

$= \frac{-2\pi ik - 2}{(\pi ik)^2}$

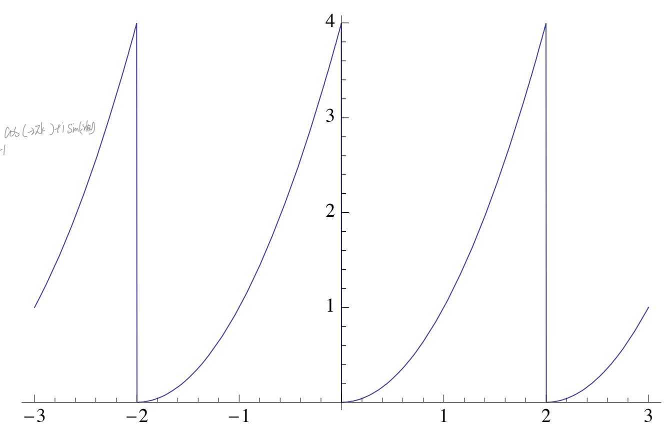
$= \frac{2(\pi ik+1)}{\pi^2 k^2}$

$\hat{f}(0) = \frac{1}{2}\int_0^2 t^2 \cdot 1\, dt$

$= \frac{1}{2} \cdot \frac{1}{3} t^3\Big|_0^2 = \frac{8}{6} = \frac{4}{3}$

$\begin{cases} C_n = \frac{2(in\pi+1)}{\pi^2 n^2} & n \neq 0 \\ C_0 = \frac{4}{3} \end{cases}$

$e^{-2\pi ik} = \cos(-2\pi k) + i\sin(-2\pi k) = 1$



   (a) Find the Fourier series coefficients, c_n, of $f(t)$.

   (b) Using your result from part (a), obtain the following: *Hint: You might want to read section 1.14.3 on pages 54 - 57 of the course reader before trying this part.*

F.S converges to the average at jump discontinuity

b) from (a). $f(t) = \sum_{n=-\infty}^{+\infty} C_n \cdot e^{\pi int}$

$\begin{cases} C_n = \frac{2(in\pi+1)}{\pi^2 n^2} \\ C_0 = \frac{4}{3} \end{cases}$

① F.S converges to average at discontinuity

at $t=0$ $\quad \sum_{k=-\infty}^{+\infty} e^{\pi ik0} \cdot C_n$

$= \frac{4}{3} + \sum_{n=1}^{+\infty} C_n + C_{-n}$

$= \frac{4}{3} + \sum_{n=1}^{+\infty} \frac{4}{\pi^2 n^2}$

$= \frac{4}{3} + \frac{4}{\pi^2} \cdot \sum_{n=1}^{+\infty} \frac{1}{n^2} = 2$

$\therefore \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

② $f(t) = \sum_{n=-\infty}^{+\infty} C_n \cdot e^{\pi int}$

$f(1) = \frac{4}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{2(in\pi+1)}{\pi^2 n^2} \cdot (\cos n\pi + i\sin n\pi)$

$= \frac{4}{3} + \sum_{n=1}^{+\infty} \frac{4}{\pi^2 n^2} \cos n\pi$

$= \frac{4}{3} + \frac{4}{\pi^2} \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} = 1$

$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$

$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^3}{12}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

$\frac{4+2}{24}$

$\frac{6}{24}$

③.

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ $\quad$ even terms + odd terms

(2) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ $\quad$ - even terms + odd terms

(1) + (2)

$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{1}{2}\left(\frac{\pi^2}{6} + \frac{\pi^2}{12}\right) = \frac{\pi^2}{8}$

2. (10 points) *Whither Rayleigh?* What happens to Rayleigh's identity if $f(t)$ is periodic of period $T \neq 1$?

Rayleigh's Identity:

$\int_0^1 |f(t)|^2\, dt = \sum_{k=-\infty}^{+\infty} |\hat{f}(k)|^2$

when $f(t)$ has period T

$g(t) = f(tT)$ has period 1

lhs $= \int_0^1 |f(tT)|^2\, dt$

$(tT = u)$

$= \int_0^T |f(u)|^2\, d\frac{u}{T}$

$= \frac{1}{T}\int_0^T |f(u)|^2\, du$

rhs $= \sum_{k=-\infty}^{+\infty} |\hat{g}(k)|^2$

$\hat{g}(k) = \int_{-\infty}^{+\infty} g(t)\, e^{-2\pi ikt}\, dt$

$= \int_{-\infty}^{+\infty} f(tT)\, e^{-2\pi ikt}\, dt$

$u = tT$

$= \int_{-\infty}^{+\infty} f(u)\, e^{-2\pi ik\, u/T}\, d\, u/T$

$= \frac{1}{T}\int_{-\infty}^{+\infty} f(u)\, e^{-2\pi ik\, u/T}\, du$

$= \frac{1}{T}\hat{f}(k/T)$

$\frac{1}{T}\int_0^T |f(u)|^2\, du = \sum_{k=-\infty}^{+\infty} \left|\frac{1}{T}\hat{f}(k/T)\right|^2$

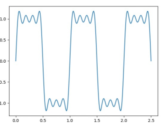
$\int_0^T |f(u)|^2\, du = \frac{1}{T}\sum_{k=-\infty}^{+\infty} |\hat{f}(k)|^2$

3. (25 points) *Sinesum2, Square Wave, High Frequency Noise and AM Modulation.*

This problem is based on the Matlab application in the 'Sinesum2 Matlab Program' section of 'Handouts' on the course website. Go there to read the directions and to get the files. It's a tool to plot sums of sinusoids of the form

$$\sum_{n=1}^{N} A_n \sin(2\pi n t + \phi_n)\,.$$

(a)

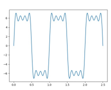


1° this function is given on the book

$$\sum_{k=0}^{4} \frac{1}{2k+1} \sin (2k+1) 2\pi t$$

2° scale

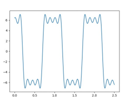
$$\sum_{k=0}^{4} \frac{6}{2k+1} \sin (2k+1) 2\pi t$$

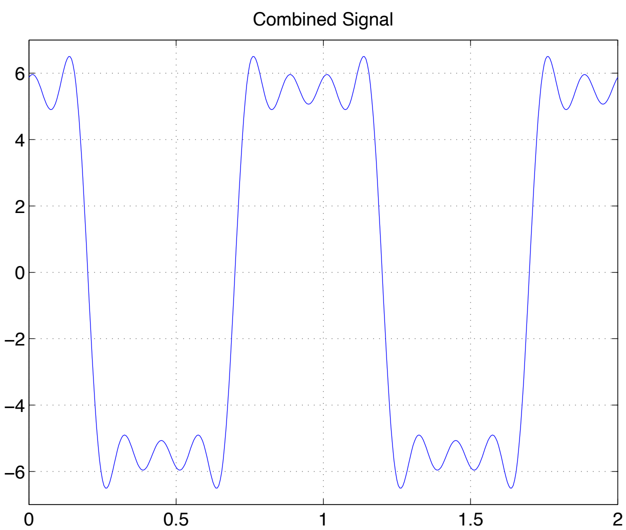


3° shift to left by 0.3

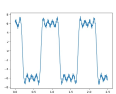
$$\sum_{k=0}^{4} \frac{6}{2k+1} \sin\left[(2k+1) 2\pi (t-0.3)\right]$$

$$= \sum_{k=0}^{4} \frac{6}{2k+1} \sin\left[(2k+1) 2\pi t - 0.3 \cdot 2 \cdot (2k+1)\pi\right]$$

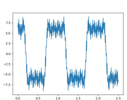




(b)



(a) + 0.5 sin(2π50t)



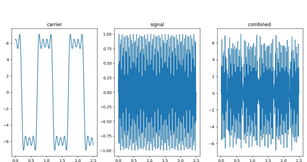
(a) + 2 sin(2π50t)

(a) Using the sinesum2 application, generate (approximately) the signal plotted above. You should be able to do this with seven harmonics. Along with the overall shape, the things to get right are the approximate shift and orientation of the signal. To get started, observe that the signal looks like a square wave signal. Recall that the square wave signal is studied in Section 1.7 of the course reader. However, you'll see that additional flipping and shifting need to be done. This can be accomplished by changing the phases ϕ_n (do that in two steps, say first a flip and then a shift). Explain what you're doing at each stage.

(b) Additive high frequency noise is very common when signals go through various communications systems. In this case, we will assume that the previous signal goes through some communication system, that adds high frequency noise which can be approximated as: $0.5\sin(2\pi 50t)$. How does this change the original signal in the time domain? What happens if the amplitude of the noise increases from 0.5 to 2. Plot your results and explain what you observe.

(c) Amplitude Modulation (AM) is a technique used in communication systems for transmitting information. Typically, AM works by varying the amplitude of a carrier signal (simple sine signal of the form $A\sin(2\pi f_c t + \phi_c)$) in relation to the information signal that needs to be transmitted. For example, if we denote the information signal we want to trasmit by $m(t)$, $m(t)A\sin(2\pi f_c t + \phi_c)$ is one type of "AM signal" (double-sideband suppressed-carrier (DSBSC) AM signal to be exact!) that we could choose to transmit. Naturally, a lot more can be said about AM, but this is not the scope of this question.

To examine an example of AM, we will assume that the signal from part (a) is multiplied by $\sin(2\pi 50t)$. Using sinesum2, plot the new signal, and explain how this can be done.



$\sin(\ )\cdot\sin(\ ) = \frac{\cos(\ ) - \cos(\ )}{2}$

(a). $f(t) = 2\cdot\Lambda_2(t-2) + 2.5\,\Lambda_2(t-4)$

$\Lambda(t) \to \mathrm{sinc}^2(s)$

$\Lambda_2(t) = \Lambda(\tfrac{1}{2}t) \to 2\,\mathrm{sinc}^2(2s)$

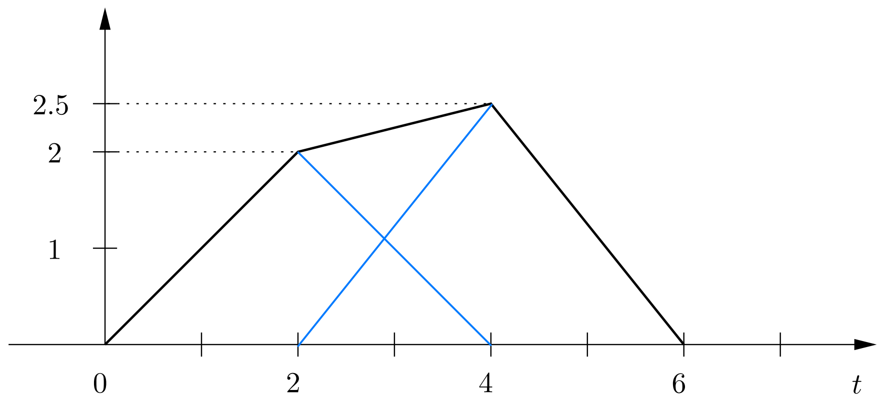
$\Lambda_2(t-2) \to 2\,\mathrm{sinc}^2(2s)\cdot e^{-4\pi i s}$ $\quad \Lambda_2(t-4) \to 2\,\mathrm{sinc}^2(2s)\, e^{-8\pi i s}$

$f(t) \to 4\,\mathrm{sinc}^2(2s)\cdot e^{-4\pi i s} + 5\,\mathrm{sinc}^2(2s)\cdot e^{-8\pi i s}$

$f(t) \to \mathrm{sinc}^2(2s)\left(4\cdot e^{-4\pi i s} + 5\,e^{-8\pi i s}\right)$

4. (20 points) *Piecewise linear approximations and Fourier transforms.*

(a) Find the Fourier transform of the following signal.



Hint: Think Λ's.

(b) Consider a signal $f(t)$ defined on an interval from 0 to D with $f(0) = 0$ and $f(D) = 0$. We get a uniform, piecewise linear approximation to $f(t)$ by dividing the interval into n equal subintervals of length $T = D/n$, and then joining the values $0 = f(0), f(T), f(2T), \ldots, f(nT) = f(D) = 0$ by consecutive line segments. Let $g(t)$ be the linear approximation of a signal $f(t)$, obtained in this manner, as illustrated in the following figure where $T = 1$ and $D = 6$.

(b)

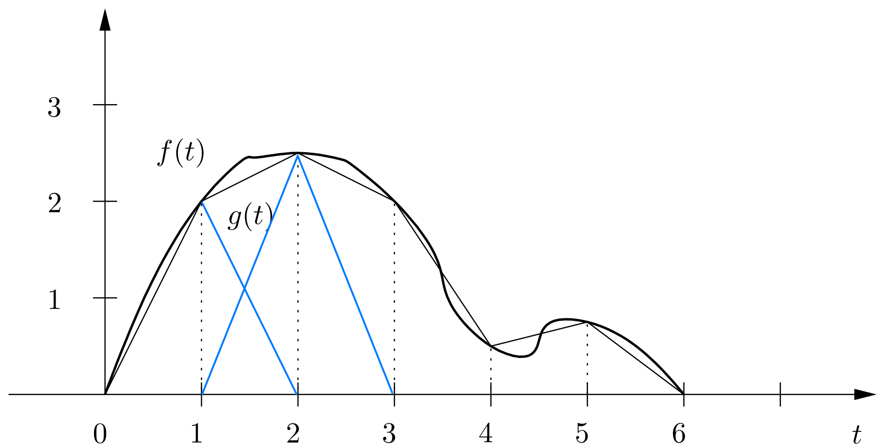
$g(t) = \sum_{x=1}^{n} f(xT)\cdot\Lambda_T(t-xT)$

$\Lambda(t) \to \mathrm{sinc}^2(s)$

$\Lambda_T(t) = \Lambda(t/T) \longrightarrow T\,\mathrm{sinc}^2(Ts)$

$\Lambda_T(t-xT) \longrightarrow T\cdot\mathrm{sinc}^2(Ts)\cdot e^{-2\pi i s xT}$

$g(t) \longrightarrow T\cdot\mathrm{sinc}^2(Ts)\cdot\sum_{x=1}^{n} f(xT)\cdot e^{-2\pi i s xT}$



Find $\mathcal{F}g(s)$ for the general problem (*not* for the example given in the figure above) using any necessary information about the signal $f(t)$ or its Fourier transform $\mathcal{F}f(s)$. Think Λ's, again.

5. (10 points) *The modulation property of the Fourier transform.*

(a). $\mathcal{F}g(s) = \int_{-\infty}^{+\infty} g(t)\, e^{-2\pi i s t}\, dt$

$= \int_{-\infty}^{+\infty} f(t)\cos(2\pi s_0 t)\, e^{-2\pi i s t}\, dt$

$= \int_{-\infty}^{+\infty} f(t)\, \frac{e^{2\pi i s_0 t} + e^{-2\pi i s_0 t}}{2}\, e^{-2\pi i s t}\, dt$

$= \frac{1}{2}\int_{-\infty}^{+\infty} f(t)\, e^{-2\pi i (s-s_0)t}\, dt + \frac{1}{2}\int_{-\infty}^{+\infty} f(t)\, e^{-2\pi i (s+s_0)t}\, dt$

$= \frac{1}{2}\mathcal{F}f(s-s_0) + \frac{1}{2}\mathcal{F}f(s+s_0)$

(a) Let $f(t)$ be a signal, s_0 a number, and define

$$g(t) = f(t)\cos(2\pi s_0 t)$$

Show that

$$\mathcal{F}g(s) = \frac{1}{2}\mathcal{F}f(s - s_0) + \frac{1}{2}\mathcal{F}f(s + s_0)$$

(No delta functions, please, for those who know about them.)

(b)

$\mathcal{F}g(s) = \Lambda_2(s+4) + \Lambda_2(s-4)$

$\operatorname{sinc}^2 t \longrightarrow \Lambda(s)$

$\operatorname{sinc}^2(2t) \longrightarrow \frac{1}{2}\Lambda(s/2) = \frac{1}{2}\Lambda_2(s)$

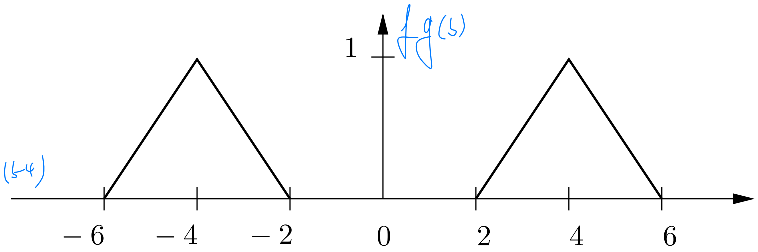
$\mathcal{F}g(s) = 2\cdot\frac{1}{2}\Lambda_2(s+4) + 2\cdot\frac{1}{2}\Lambda_2(s-4)$

$= 2\cdot\mathcal{F}\operatorname{sinc}^2(2t)(s+4) + 2\,\mathcal{F}\operatorname{sinc}^2(2t)(s-4)$

$= \frac{1}{2}\mathcal{F}\,4\operatorname{sinc}^2(2t)(s+4) + \frac{1}{2}\mathcal{F}\,4\operatorname{sinc}^2(2t)(s-4)$

$g(t) = \cos(2\pi 4t)\cdot 4\cdot\operatorname{sinc}^2(2t)$

(b) Find the signal (in the time domain) whose Fourier transform is pictured, below.



6. (10 points) *Fourier transforms and Fourier coefficients* Suppose the function $f(t)$ is zero outside the interval $-1/2 \le t \le 1/2$. We form a function $g(t)$ which is a periodic version of $f(t)$ with period 1 by the formula

$$g(t) = \sum_{k=-\infty}^{\infty} f(t-k)\,.$$

The Fourier series representation of $g(t)$ is given by

$$g(t) = \sum_{k=-\infty}^{\infty} \hat{g}(n)e^{2\pi int}\,.$$

Find the relationship between the Fourier transform $\mathcal{F}f(s)$ and the Fourier series coefficients $\hat{g}(n)$.

Fourier coefficients:

$\hat{g}(n) = \int_0^1 g(t)\, e^{-2\pi int}\, dt$

$= \int_{-1/2}^{1/2} g(t)\, e^{-2\pi int}\, dt$

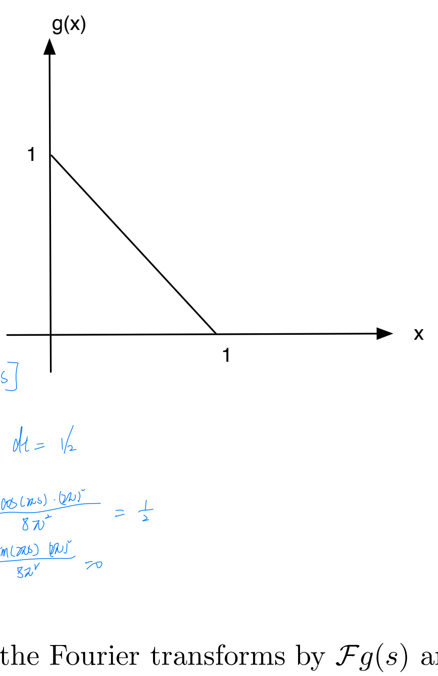
Fourier Transform

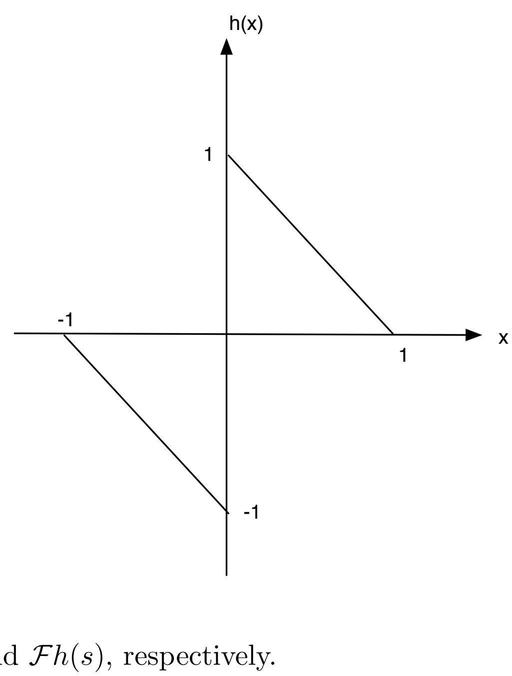
$\mathcal{F}f(s) = \int_{-\infty}^{+\infty} f(t)\, e^{-2\pi ist}\, dt$

$= \int_{-1/2}^{1/2} f(t)\, e^{-2\pi ist}\, dt$

$\therefore\ \mathcal{F}f(n) = \hat{g}(n)$

7. (50 points) Consider the functions $g(x)$ and $h(x)$, shown below





(a)

$\mathcal{F}g(s) = \int_{-\infty}^{+\infty} g(t)\, e^{-2\pi ist}\, dt$

$= \int_0^1 (-t+1)e^{-2\pi ist}\, dt$

$= \frac{1}{-2\pi is}\int_0^1 (-t+1)\, de^{-2\pi ist}$

$= \frac{1}{-2\pi is}\left[(-t+1)\, e^{-2\pi ist}\Big|_0^1 - \int_0^1 e^{-2\pi ist}\, d(-t+1)\right]$

$= \frac{1}{-2\pi is}\left[-1 + \int_0^1 e^{-2\pi ist}\, dt\right]$

$= \frac{1}{-2\pi is}\left[-1 + \frac{1}{-2\pi is} e^{-2\pi ist}\Big|_0^1\right]$

$= \frac{1}{2\pi is} - \frac{1}{(2\pi s)^2}\left[e^{-2\pi is} - 1\right]$

$= \frac{1}{2\pi is} - \frac{1}{(2\pi s)^2}\left[\cos(2\pi s) + i\sin(-2\pi s) - 1\right]$

$= \frac{1}{(2\pi s)^2}\left[-2\pi is - i\sin(-2\pi s) + 1 - \cos(-2\pi s)\right]$

$= \frac{1}{(2\pi s)^2}\left[1 - \cos(2\pi s)\right] + \frac{i}{(2\pi s)^2}\left[\sin(2\pi s) - 2\pi s\right]$

$\mathcal{F}g(0) = \int_{-\infty}^{+\infty} g(t)\, e^{-2\pi i 0 t}\, dt = \int_0^1 -t+1\, dt = 1/2$

$\lim_{s\to 0} \frac{1-\cos(2\pi s)}{(2\pi s)^2} = \lim_{s\to 0} \frac{\sin(2\pi s)\cdot 2\pi}{2\cdot 2\pi s\cdot 2\pi} = \lim_{s\to 0} \frac{\cos(2\pi s)\cdot(2\pi)^2}{8\pi^2} = \frac{1}{2}$

$\lim_{s\to 0} \frac{\sin(2\pi s) - 2\pi s}{(2\pi s)^2} = \lim_{s\to 0} \frac{\cos(2\pi s)\cdot 2\pi - 2\pi}{2\cdot 2\pi s\cdot 2\pi} = \lim_{s\to 0} \frac{-\sin(2\pi s)\,(2\pi)^2}{32^2} = 0$

$\therefore\ \lim_{s\to 0}\mathcal{F}g(s) = \mathcal{F}g(0)$

$\therefore\ \mathcal{F}g(s)$ is continuous at 0

Denote the Fourier transforms by $\mathcal{F}g(s)$ and $\mathcal{F}h(s)$, respectively.

(a) (5 points) Lykomidis says that the imaginary part of $\mathcal{F}g(s)$ is $(\sin(2\pi s) - 2\pi s)\,/\,(4\pi^2 s^2)$. Brad, however, expresses concerns about Lykomidis' work. He is not sure that this can

(b) from (a)

$$\mathcal{F}g(s) = \frac{1}{(2\pi s)^2}\left[1-\cos 2\pi s\right] + \frac{i}{(2\pi s)^2}\left[\sin 2\pi s - 2\pi s\right]$$

$h(t) = g(t) + -g(-t)$

$\mathcal{F}h(s) = \mathcal{F}g(s) - \mathcal{F}g(-s)$

$= \frac{2i}{(2\pi s)^2}\left[\sin 2\pi s - 2\pi s\right]$

$\mathcal{F}h(s)$ has only imaginary pars

$\mathcal{F}h(s) = e^{i\theta} = \cos\theta + i\sin\theta$

to make $\cos\theta = 0$, $\theta = \pm\frac{\pi}{2}$

be the imaginary part of g. "Why would it have a singularity at $s = 0$?" Brad says, as a general fact, if a function is integrable, as $g(t)$ is, then its Fourier transform is continuous. Lykomidis, asks Brad whether he is willing to buy him coffee if he (Lykomidis) can prove him (Brad) wrong. Brad feels very confident and quickly accepts. Is Lykomidis getting free coffee?

(b) (5 points) What are the two possible values of $\angle\mathcal{F}h(s)$, i.e., the phase of $\mathcal{F}h(s)$? Express your answer in radians.

(c) (5 points) Evaluate $\int_{-\infty}^{\infty} \mathcal{F}g(s)\cos(\pi s)\,ds$.

(d) (5 points) Evaluate $\int_{-\infty}^{\infty} \mathcal{F}h(s)e^{i4\pi s}\,ds$.

(e) (10 points) Without performing any integration, what is the real part of $\mathcal{F}g(s)$? Explain your reasoning.

(f) (10 points) Without performing any integration, what is $\mathcal{F}h(s)$? Explain your reasoning.

(g) (10 points) Suppose $h(x)$ is periodized to have period $T = 2$. Without performing any integration, what are the Fourier coefficents, c_k, of this periodic signal?

(c).

from (a)

$$\mathcal{F}g(s) = \frac{1}{(2\pi s)^2}\left[1-\cos 2\pi s\right] + \frac{i}{(2\pi s)^2}\left[\sin 2\pi s - 2\pi s\right]$$

$$\int_{-\infty}^{+\infty} \mathcal{F}g(s)\cdot\cos(\pi s)\,ds$$

$$= \int_{-\infty}^{+\infty} \mathcal{F}g(s)\cdot\frac{1}{2}\left(e^{i\pi s} + e^{-i\pi s}\right)ds$$

$$= \frac{1}{2}\underbrace{\int_{-\infty}^{+\infty} \mathcal{F}g(s)\,e^{i\pi s}\,ds}_{①} + \frac{1}{2}\int_{-\infty}^{+\infty} \mathcal{F}g(s)\,e^{-i\pi s}\,ds$$

① $\int_{-\infty}^{+\infty} \mathcal{F}g(s)\,e^{i\pi s}\,ds$

$= \int_{-\infty}^{+\infty} \mathcal{F}g(s)\,e^{2\pi i s\cdot\frac{1}{2}}\,ds$

$= (\mathcal{F}^{-1}\mathcal{F}g)(\frac{1}{2}) = g(1/2) = 1/2$

② $\int_{-\infty}^{+\infty} \mathcal{F}g(s)\,e^{-i\pi s}\,ds$

$= \int_{-\infty}^{+\infty} \mathcal{F}g(s)\,e^{2\pi i s(-1/2)}\,ds$

$= (\mathcal{F}^{-1}\mathcal{F}g)(-1/2) = g(-1/2) = 0$

$= \frac{1}{2}\cdot\frac{1}{2} = 1/4$

(d). $\int_{-\infty}^{+\infty} \mathcal{F}h(s)\,e^{i4\pi s}$

$= \int_{-\infty}^{+\infty} \mathcal{F}h(s)\cdot e^{2\pi i s 2}\,ds$

$= (\mathcal{F}^{-1}\mathcal{F}h)(2)$

$= h(2) = 0$