# EE 261 The Fourier Transform and its Applications
# Fall 2007
# Problem Set One     Due Wednesday, October 3

1. *Some practice with geometric sums and complex exponentials*

   We'll make much use of formulas for the sum of a geometric series, especially in combination with complex exponentials.

   (a) If $w$ is a real or complex number, $w \neq 1$, and $p$ and $q$ are any integers, show that

   $$\sum_{n=p}^{q} w^n = \frac{w^p - w^{q+1}}{1-w}\,.$$

   (Of course if $w = 1$ then the sum is $\sum_{n=p}^{q} 1 = q + 1 - p$.)
   Discuss the cases when $p = -\infty$ or $q = \infty$. What about $p = -\infty$ *and* $q = +\infty$?

   a). $\sum_{n=p}^{q} w^n = w^p + w^{p+1} \cdots + w^q = w^p\left(\frac{1-w^{q-p+1}}{1-w}\right) = \frac{w^p - w^{q+1}}{1-w}$ $(p \neq -\infty\ q \neq \infty)$

   when $p = -\infty$ $\lim_{p\to-\infty} \frac{w^p - w^{q+1}}{1-w} = \begin{cases} \frac{-w^{q+1}}{1-w} & |w|>1 \\ \text{does not converge} \end{cases}$

   when $q = \infty$ $\lim_{q\to\infty} \frac{w^p - w^{q+1}}{1-w} = \begin{cases} \frac{w^p}{1-w} & |w|<1 \\ \text{does not converge} \end{cases}$

   (b) Find the sum

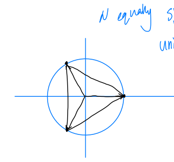
   $$\sum_{n=0}^{N-1} e^{2\pi i n/N}$$

   and explain your answer geometrically.

   b). $\sum_{n=0}^{N-1} e^{2\pi i n/N} = \sum_{n=0}^{N-1} \left[e^{2\pi i/N}\right]^n = \frac{1-e^{2\pi i/N \cdot N}}{1-e^{2\pi i/N}} = \frac{1-e^{2\pi i}}{1-e^{2\pi i/N}} = \frac{1-(\cos 2\pi + i\sin 2\pi)}{1-e^{2\pi i/N}} = 0$

   $\sum_{n=0}^{N-1} e^{2\pi i n/N}$ $e^{2\pi i n/N} = \cos n/N \cdot 2\pi + i\sin n/N \cdot 2\pi$

   N equally spaced points on the unit circle

   (c) Derive the formula

   $$\sum_{k=-N}^{N} e^{2\pi i k t} = \frac{\sin(2\pi t(N+1/2))}{\sin(\pi t)}$$

   c). $\sum_{k=-N}^{N} e^{2\pi i k t} = \frac{1-(e^{2\pi i t})^{2N+1}}{1-e^{2\pi i t}} \cdot e^{-2\pi i N t} = \frac{e^{-2\pi i N t} - e^{2\pi i (N+1)t}}{1-e^{2\pi i t}}$

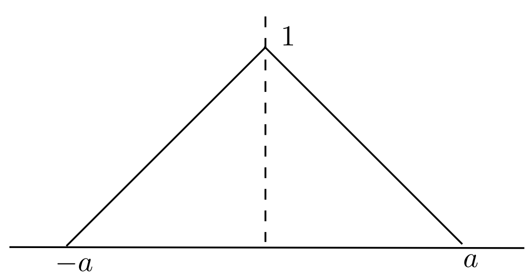
   $= \frac{e^{\pi i t}}{e^{\pi i t}} \cdot \frac{e^{-2\pi i (N+1/2)t} - e^{2\pi i (N+1/2)t}}{e^{-\pi i t} - e^{\pi i t}} = \frac{-2\sin(-2\pi(N+1/2)t)}{-2\sin \pi t} = \frac{\sin(2\pi t(N+1/2))}{\sin \pi t}$

2. *Some practice combining simple signals.* (5 points each)

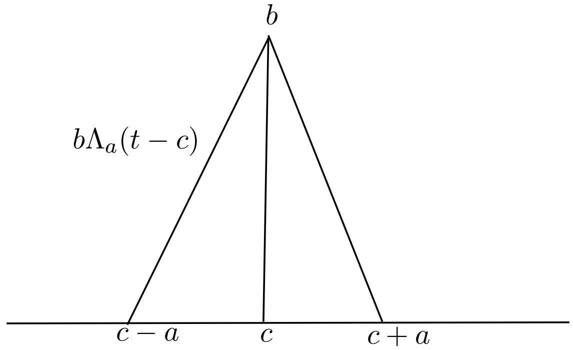
   The triangle function with a parameter $a > 0$ is

   $$\Lambda_a(t) = \Lambda(t/a) = \begin{cases} 1 - \frac{1}{a}|t|\,, & |t| \leq a \\ 0\,, & |t| > a \end{cases}$$

   The graph is



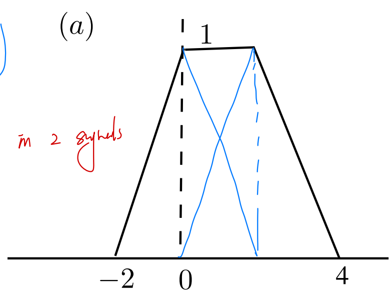
The parameter $a$ specifies the width, namely $2a$. Alternately, $a$ determines the slopes of the sides: the left side has slope $1/a$ and the right side has slope $-1/a$. We can modify $\Lambda_a$ by scaling the height and shifting horizontally, forming $b\Lambda_a(t-c)$. The slopes of the sides of the scaled function are then $\pm b/a$. The graph is:

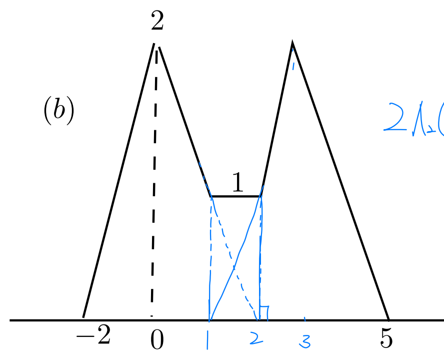



Express each of the following as a sum of two shifted, scaled triangle functions $b_1\Lambda_{a_1}(t-c_1) + b_2\Lambda_{a_2}(t-c_2)$. Think of the sum as 'left-triangle' plus a 'right-triangle' ('right' meaning to the right, not having an angle of 90°). For part (d), the values $x_1$, $x_2$ and $x_3$ cannot be arbitrary. Rather, to be able to express the plot as the sum of two $\Lambda$'s they must satisfy a relationship that you should determine.

$\Lambda_2(t) + \Lambda_2(t-2)$

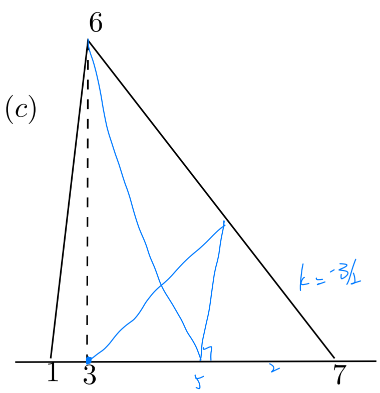
turning point → peak/dieoff in 2 signals



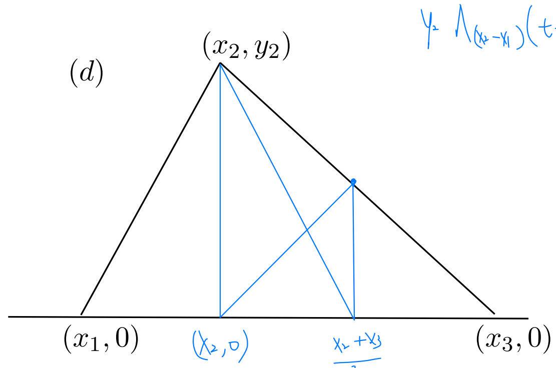




$2\Lambda_2(t) + 2\Lambda_2(t-3)$

$6\Lambda_2(t-3) + 3\Lambda_2(t-5)$



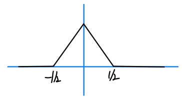

$y_2\Lambda_{(x_2-x_1)}(t-x_2) + \frac{y_2}{2}\Lambda_{(\frac{x_3-x_2}{2})}(t-\frac{x_2+x_3}{2})$



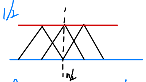

3. *Creating periodic functions.* (5 points each)

a) $g(t+T) = \sum_{n=-\infty}^{+\infty} f(t+T-nT)$
$= \sum_{n=-\infty}^{+\infty} f(t-(n-1)T)$
$= \sum_{k=-\infty}^{+\infty} f(t-kT)$
$= g(t)$

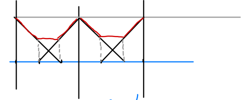
b)



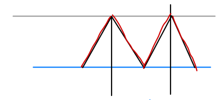
when $T=1/2$



each hat function overlaps with its left and right neighbor
(piecewise linear, just connect end points)



$T=3/4$



$t=1$

Let $f(t)$ be a function, defined for all $t$, and let $T > 0$. Define

c) no.
suppose $f(t)$ has period $a$
take $T=a$
$\sum_{n=-\infty}^{+\infty} f(t-na)$
$= \sum_{n=-\infty}^{+\infty} f(t)$
does not even converge when $f(t) \neq 0$

$$g(t) = \sum_{n=-\infty}^{\infty} f(t - nT).$$

(a) Provided the sum converges, show that $g(t)$ is periodic with period $T$. One sometimes says that $g(t)$ is the periodization of $f(t)$.

(b) Let $f(t) = \Lambda_{1/2}(t)$. Sketch the periodizations $g(t)$ of $f(t)$ for $T = 1/2$, $T = 3/4$, $T = 1$, $T = 2$. (analogy from 18.085 finite elements trial function)

(c) If a function $f(t)$ is already periodic, is it equal to its own periodization? Explain.

4. *Combining periodic functions.* (5 points each)

a) yes
$\sin(2\pi mx)$ has period $1/m$
$\sin(2\pi nx)$ has period $1/n$
$f(x)$ has period $T$ such that
$T = \min\{t \mid t \% m = 0,\ t \% n = 0\}$

(a) Let $f(x) = \sin(2\pi mx) + \sin(2\pi nx)$ where $n$ and $m$ are positive integers. Is $f(x)$ periodic? If so, what is its period?

(b) Let $g(x) = \sin(2\pi px) + \sin(2\pi qx)$ where $p$ and $q$ are positive rational numbers (say $p = m/r$ and $q = n/s$, as fractions in lowest terms). Is $g(x)$ periodic? If so, what is its period? from (a)

c). when $t=0$ $f(t)=2$
$\cos t + \cos\sqrt{2}t = 2\cos\frac{(1+\sqrt{2})t}{2}\cos\frac{(1-\sqrt{2})t}{2} := 2$
$\cos\frac{(1+\sqrt{2})t}{2} \cdot \cos\frac{(1-\sqrt{2})t}{2} = 1$
$\because \cos(\frac{1+\sqrt{2}}{2})t \in [-1,1]$
$\cos(\frac{1-\sqrt{2}}{2})t \in [-1,1]$
$\therefore$ ① $\cos\frac{(1+\sqrt{2})t}{2} = \cos\frac{(1-\sqrt{2})t}{2} = 1$
② $\cos(\frac{1+\sqrt{2}}{2})t = \frac{\cos(1-\sqrt{2})t}{2} = -1$
if $(\frac{1+\sqrt{2}}{2})t = k\pi$
then $\frac{1-\sqrt{2}}{2}t = -k\pi - t$
$\cos t + \cos\sqrt{2}t = 2$ only when $t=0$

(c) Show that $f(t) = \cos t + cos\sqrt{2}t$ is not periodic. (Hint: Suppose by way of contradiction that there is some $T$ such that $f(t + T) = f(t)$ for all $t$. In particular, the maximum value of $f(t)$ repeats. This will lead to a contradiction.)

(d) Consider the voltage $v(t) = 3\cos(2\pi\nu_1 t - 1.3) + 5\cos(2\pi\nu_2 t + 0.5)$. Regardless of the frequencies $\nu_1$, $\nu_2$ the maximum voltage is always less than 8, but it can be much smaller. Use MATLAB (or another program) to find the maximum voltage if $\nu_1 = 2$ Hz and $\nu_2 = 1$ Hz. [From Paul Nahim, *The Science of Radio*]

5. *Some practice with inner products.* (5 points each)

what matters is the "relative position" between 2 signals

Let $f(t)$ and $g(t)$ be two signals with inner product

$$(f, g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt\,.$$

some kind of "affine-invariance"

Define the *reversed signal* to be

$$f^{-}(t) = f(-t)\,.$$

Define the *delay operator*, or *shift operator* by

$$\tau_a f(t) = f(t - a)\,.$$

a) $\int_{-\infty}^{+\infty} f(-t)\overline{g(-t)}\,dt$
$= \int_{+\infty}^{-\infty} f(s)\overline{g(s)}\,d{-s}$
$= \int_{-\infty}^{+\infty} f(s)\overline{g(s)}\,ds$
does not change

b) mess up.

c). $\int_{-\infty}^{+\infty} f(t-a)\overline{g(t-a)}\,dt$
$= \int_{-\infty}^{+\infty} f(t-a)\overline{g(t-a)}\,d(t-a)$
$= \int_{-\infty}^{+\infty} f(s)\overline{g(s)}\,ds$
does not change

(a) If both $f(t)$ and $g(t)$ are reversed, what happens to their inner product?

(b) If one of $f(t)$ and $g(t)$ is reversed, what happens to their inner product?

(c) If both $f(t)$ and $g(t)$ are shifted by the same amount, what happens to their inner product?

(d) If one of $f(t)$ and $g(t)$ is shifted, what happens to their inner product?

(e) If both $f(t)$ and $g(t)$ are shifted but by different amounts, what happens to their inner product?

d) mess up
$(\tau_a f, g) = (f, \tau_{-a} g)$

e). same argument

(f), if the relative position is changed by a multiple of 1 the inner product does not change

(f) What, if anything, changes in these results if $f$ and $g$ are periodic of period 1 and their inner product is

$$(f\,,\,g) = \int_0^1 f(t)\overline{g(t)}\,dt\,.$$

6. *The Dirichlet Problem, Convolution, and the Poisson Kernel* When modeling physical phenomena by partial differential equations it is frequently necessary to solve a *boundary value problem.* One of the most famous and important of these is associated with Laplace's equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,,$$

where $u(x, y)$ is defined on a region $R$ in the plane. The operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is called the *Laplacian* and a real-valued function $u(x, y)$ satisfying $\Delta u = 0$ is called *harmonic.* The *Dirichlet problem* for Laplace's equation is this:

> Given a function $f(x, y)$ defined on the boundary of a region $R$, find a function $u(x, y)$ defined on $R$ that is harmonic and equal to $f(x, y)$ on the boundary.

Fourier series and convolution combine to solve this problem when $R$ is a disk. As with many problems with circular symmetry is helpful to introduce polar coordinates $(r, \theta)$, with $x = r\cos\theta$, $y = r\sin\theta$. Writing $U(r, \theta) = u(r\cos\theta, r\sin\theta)$, so regarding $u$ as a function of $r$ and $\theta$, Laplace's equation becomes

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0\,. \qquad (1)$$

(You need not derive this.)

(a). $\frac{\partial U}{\partial r} = 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} C_n \cdot n \cdot r^{n-1} e^{in\theta}\right\}$

$\frac{\partial^2 U}{\partial r^2} = 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} C_n \cdot n(n-1) r^{n-2} e^{in\theta}\right\}$

$\frac{\partial U}{\partial \theta} = 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} C_n r^n \cdot e^{in\theta} \cdot in\right\}$

$\frac{\partial^2 U}{\partial \theta^2} = -2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} C_n r^n \cdot e^{in\theta} \cdot n^2\right\}$

$\left[C_n \cdot n \cdot (n-1) \cdot r^{n-2} \cdot e^{in\theta}\right]^2 + \frac{1}{r}\left[C_n \cdot n \cdot r^{n-1} \cdot e^{in\theta}\right] + 1/r^2\left[C_n r^n \cdot e^{in\theta} \cdot n^2\right] = 0$

$\therefore U(r,\theta) = \mathrm{Re}\left\{C_0 + 2\sum_{n=1}^{\infty} C_n r^n e^{in\theta}\right\}$ satisfies (1)

Thus $U(r,\theta)$ is harmonic

actually any $f(x+iy)$ satisfies the laplace equation

(a) Let $\{c_n\}$, $n = 0, 1, \ldots$ be a bounded sequence of complex numbers, let $r < 1$ and define $u(r, \theta)$ by the series

$$u(r, \theta) = \mathrm{Re}\left\{c_0 + 2\sum_{n=1}^{\infty} c_n r^n e^{in\theta}\right\}\,.$$

From the assumption that the coefficients are bounded, and comparison with a geometric series, it can be shown that the series converges. Show that $u(r, \theta)$ is a harmonic function.

(b) Suppose that $f(\theta)$ is a real-valued, continuous, periodic function of period $2\pi$ and let

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

be its Fourier series. Now form the harmonic function $u(r, \theta)$ as above, with these coefficients $c_n$. This solves the Dirichlet problem of finding a harmonic function on the unit disk $x^2 + y^2 < 1$ with boundary values $f(\theta)$ on the unit circle $x^2 + y^2 = 1$; precisely,

$$\lim_{r \to 1} u(r, \theta) = f(\theta)\,.$$

(b) $f(\theta) = \sum_{n=-\infty}^{+\infty} C_n \cdot e^{in\theta}$

$U(r,\theta) = \mathrm{Re}\left\{C_0 + 2\sum_{n=1}^{\infty} C_n \cdot r^n \cdot e^{in\theta}\right\}$

$= \frac{1}{2}\left\{C_0 + \overline{C_0} + \sum_{n=1}^{\infty} C_n r^n \cdot e^{in\theta} + \sum_{n=1}^{\infty} \overline{C_n}\, r^n e^{-in\theta}\right\}$

$\because$ $C_n$ are Fourier coefficients of $f$

$\therefore$ $C_0$ is real, $C_n = \overline{C_{-n}}$ for $n \neq 0$

$U(r,\theta) = \sum_{n=-\infty}^{+\infty} C_n \cdot r^n e^{in\theta}$

$U(1,\theta) = f(\theta)$

$\therefore U(r,\theta) = \mathrm{Re}\left\{C_0 + 2\sum_{n=1}^{\infty} C_n r^n \cdot e^{in\theta}\right\}$ solves the laplace equation with boundary condition $U(1,\theta) = f(\theta)$ where $U(1,\theta) = f(\theta)$ has Fourier Series $f(\theta) = \sum_{n=-\infty}^{+\infty} C_n \cdot e^{in\theta}$



18.085 Lec 24.

You are not asked to show this – it requires a fair amount of work – but assuming that all is well with convergence, explain why one has

$$u(1,\theta) = f(\theta)\,.$$

[This uses the symmetry property of Fourier coefficients.]

(c) The solution can also be written as a convolution: show that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)P(r,\theta-\phi)\,d\phi\,,$$

where

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}\,.$$

[Introduce the Fourier coefficients of $f$. You'll have to sum a geometric series.]

(d) The function $P(r,\theta)$ is called the *Poisson kernel*. Show that it is a harmonic function. [This is a special case of the result you established in Part (a).]

Parts (c) and (d) together exhibit a property of convolution that we'll see repeatedly; The convolution of two functions inherits the nicest properties of each. In this case we convolve a continuous function $f$ (a good property) with a harmonic function $P$ (a nicer property) and the result is a harmonic function.

(c).

in (b), it's shown that

$$u(r,\theta) = \sum_{n=-\infty}^{+\infty} c_n r^n e^{in\theta}$$

where $u(1,\theta) = f(\theta) = \sum_{n=-\infty}^{+\infty} c_n e^{in\theta}$

$$\therefore c_n = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, e^{-in\phi}\, d\phi$$

$$\therefore u(r,\theta) = \sum_{n=-\infty}^{+\infty} \int_0^{2\pi} f(\phi)\, e^{-in\phi}\, d\phi \cdot r^n \cdot e^{in\theta}$$

$$= \int_0^{2\pi} \left[\sum_{n=-\infty}^{+\infty} e^{in(\theta-\phi)} \cdot r^n\right] f(\phi)\, d\phi$$

$$P(r,\theta) = \sum_{n=-\infty}^{+\infty} e^{in\theta} \cdot r^n$$

$$= 1 + 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} e^{in\theta}\cdot r^n\right\}$$

$$= 1+2\,\mathrm{Re}\left\{\frac{r e^{i\theta}(1-0)}{1-r\cdot e^{i\theta}}\right\}$$

$$= 1+2\,\mathrm{Re}\left\{\frac{r(\cos\theta + i\sin\theta)}{1-r(\cos\theta+i\sin\theta)}\right\}$$

$$= 1+2\,\frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2}$$

$$= \frac{1-2r\cos\theta+r^2+2r\cos\theta-2r^2}{1-2r\cos\theta+r^2}$$

$$= \frac{1-r^2}{1-2r\cos\theta+r^2}$$

$$\frac{r(\cos\theta+i\sin\theta)}{1-r\cos\theta - ir\sin\theta}$$

$$= \frac{r(\cos\theta+i\sin\theta)(1-r\cos\theta+ir\sin\theta)}{(1-r\cos\theta-ir\sin\theta)(1-r\cos\theta+ir\sin\theta)}$$

$$= \frac{(r\cos\theta+ir\sin\theta)(1-r\cos\theta+ir\sin\theta)}{(1-r\cos\theta)^2+r^2\sin^2\theta}$$

(d) from (c)

$$P(r,\theta) = \mathrm{Re}\left\{1+2\sum_{n=1}^{\infty} e^{in\theta}\cdot r^n\right\}$$