eg filtering

$$f(t)$$
$$\downarrow \mathcal{F}$$
$$F(s)$$
$$\downarrow \text{cut off high frequency by multiplying a rectangle function}$$
$$\Pi F(s)$$
$$\downarrow$$
$$\mathcal{F}^{-1}\Pi \cdot F = \mathcal{F}^{-1}(\Pi \cdot \mathcal{F}f)$$
$$\downarrow \text{multiplication in frequency domain}$$
$$\mathcal{F}^{-1}\Pi * f = \text{sinc} * f$$

low-pass filter

Filtering is often synonymous with convolution

Filter is a system that convolves an input with a fixed function (impulse response)

time domain $\underbrace{g}_{\text{output}} = \underbrace{f}_{\text{input}} * \underbrace{h}_{\text{impulse response}}$

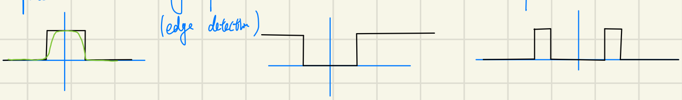
frequency domain $G(s) = \underbrace{F(s)}_{\mathcal{F}f} \cdot \underbrace{H(s)}_{\text{transfer function}}$

design filter $\rightarrow$ design transfer function

low pass

high pass (edge detection)

band pass



· Interpret Convolution

In many context, convolution is associated with smoothing / averaging

In general $f * g$ has the best properties of f and g separately

eg. $\Pi * \Pi = \Lambda$

the rectangle function is discontinuous and the triangle function is continuous

f is differentiable and g is not, $f * g$ is differentiable and $(f*g)' = f' * g$

$(f*g)(x) = \int_{-\infty}^{+\infty} f(x-y)\,g(y)\,dy$ thus similar holds for higher derivative

$\frac{d}{dx}(f*g)(x) = \int_{-\infty}^{+\infty} f'(x-y)\,g(y)\,dy = f' * g$

Convolution and Differential Equation

derivative theorem for Fourier Transform

$\mathcal{F}f'(s) = 2\pi i s \cdot \mathcal{F}f(s)$ under some technical assumption

$$\int_{-\infty}^{+\infty} f'(t)\, e^{-2\pi i s t}\, dt = \int_{-\infty}^{+\infty} e^{-2\pi i s t}\, d f(t) = e^{-2\pi i s t} f(t)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(t)\, d e^{-2\pi i s t}$$ (when $f(t) \to 0$ as $t \to \pm\infty$)

Fourier transform turns a differentiation into multiplication

$\mathcal{F}f^{(n)}(s) = (2\pi i s)^n\, \mathcal{F}f(s)$

Heat equation on a infinite rod

$u(x,t)$ is the temperature at position x at time t

$u(x,0) = f(x)$

Heat equation $\frac{du}{dt} = \frac{1}{2}\frac{d^2u}{dx^2}$

take F.T in space variable $u(x,t) \to U(s,t)$

$$\mathcal{F}\frac{du}{dt}(s,t) = \int_{-\infty}^{+\infty} \frac{d}{dt} u(x,t)\, e^{-2\pi i s x}\, dx$$

$$= \frac{d}{dt}\int_{-\infty}^{+\infty} e^{-2\pi i s t}\, u(x,t)\, dx$$

$$= \frac{d}{dt} U(s,t)$$

$$\mathcal{F}\frac{d^2u}{dx^2}(s,t) = (2\pi i s)^2 \cdot U(s,t)$$

$$= -4\pi^2 s^2\, U(s,t)$$

$\to$ $\frac{d}{dt}U(s,t) = -4\pi^2 s^2\, U(s,t)$

$$U(s,t) = e^{-4\pi^2 s^2 t} \cdot U(s,0)$$

$$U(s,0) = \int_{-\infty}^{+\infty} u(x,0) \cdot e^{-2\pi i s t}\, dt = F(s)$$

$U(s,t) = F(s) \cdot e^{-4\pi^2 s^2 t}$ product of 2 functions

$$u(x,t) = f(x) * \mathcal{F}^{-1} e^{-4\pi^2 s^2 t}$$

$$= f(x) * \frac{1}{\sqrt{2\pi t}}\, e^{-x^2/2t}$$

convolution of the initial condition with the heat kernel