· Delay

a signal is delayed / shifted

$$\int_{-\infty}^{+\infty} f(t-b)\, e^{-2\pi i s t}\, dt \qquad u := t-b$$

$$= \int_{-\infty}^{+\infty} f(u)\, e^{-2\pi i s (u+b)}\, du$$

$$= e^{-2\pi i s b} \int_{-\infty}^{+\infty} f(u)\, e^{-2\pi i s u} \cdot du$$

$$= e^{-2\pi i s b}\, F(s)$$

$f(t) \rightarrow F(s)$

$f(t-b) \rightarrow e^{-2\pi i s b}\, F(s)$

a shift in time corresponds to a phase shift in frequency

$F(s)$ is a complex number

$F(s) = |F(s)| \cdot e^{2\pi i\, \theta(s)}$

$F(s) \cdot e^{-2\pi i s b} = |F(s)|\, e^{2\pi i (\theta(s) - sb)}$ → magnitude stays the same

· Stretching

$$\int_{-\infty}^{+\infty} f(at) \cdot e^{-2\pi i s t}\, dt \qquad (a>0)$$

$$\overset{u=at}{=} \int_{-\infty}^{+\infty} f(u) \cdot e^{-2\pi i s\, u/a}\, d\frac{u}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{+\infty} f(u)\, e^{-2\pi i\, u s/a}\, du$$

$$= \frac{1}{a}\, F(s/a)$$

$$\int_{-\infty}^{+\infty} f(at)\, e^{-2\pi i s t}\, dt \qquad a<0$$

$$\overset{u=at}{=} \int_{+\infty}^{-\infty} f(u)\, e^{-2\pi i s\, u/a}\, d\, u/a$$

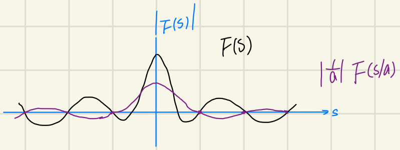
$$= -\frac{1}{a} \int_{-\infty}^{+\infty} f(u)\, e^{-2\pi i\, u\, s/a}\, du$$

$$= -\frac{1}{a}\, F(s/a)$$

$f(t) \rightarrow F(s)$

$f(at) \rightarrow \left|\frac{1}{a}\right| F(s/a)$

$a > 1$: the signal shrinks



the spectrum squeezes vertically and streched horizontally

when $a \to +\infty$ the signal gets compressed / localized in time } reciprocal relationship
the frequency spreads out

signal cannot be localized in time and in frequency (Heisenberg uncertainty principle)

the gaussian $e^{-\pi t^2} \rightarrow e^{-\pi s^2}$: the gaussian is "perfectly balanced" in both time and frequency

· Convolution

probably the most frequently used technique in $\underbrace{\text{signal processing}}$

use one function to modify another (more common in frequency domain)
(modify the spectrum of a signal)

eg. linearity $\mathcal{F}(f+g) = \mathcal{F}f + \mathcal{F}g$

modify the spectrum of f by adding the spectrum of g

multiply: $\mathcal{F}f(s) \cdot \mathcal{F}g(s) = (\mathcal{F}?)(s)$?

$$\begin{aligned}
\mathcal{F}g(s) \cdot \mathcal{F}f(s) &= \int_{-\infty}^{+\infty} g(t)\, e^{-2\pi i s t}\, dt \cdot \int_{-\infty}^{+\infty} f(x) \cdot e^{-2\pi i s x}\, dx \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-2\pi i s t} \cdot e^{-2\pi i s x}\, g(t)\, f(x)\, dt\, dx \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-2\pi i s (t+x)}\, g(t)\, f(x)\, dt\, dx \\
&= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} e^{-2\pi i s (t+x)}\, g(t)\, dt \right] f(x)\, dx \\
&\overset{u=t+x}{=} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} e^{-2\pi i s u}\, g(u-x)\, du \right] f(x)\, dx \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-2\pi i s u}\, g(u-x)\, f(x)\, du\, dx \\
&= \int_{-\infty}^{+\infty} \underbrace{\left[\int_{-\infty}^{+\infty} g(u-x)\, f(x)\, dx \right]}_{h(u) = \int_{-\infty}^{+\infty} g(u-x) f(x) dx \text{ the convolution}} e^{-2\pi i s u}\, du \\
&= \int_{-\infty}^{+\infty} (f*g)(u)\, e^{-2\pi i s u}\, du \\
&= \mathcal{F}(f*g)(s)
\end{aligned}$$

define the convolution of f and g

$$(f*g)(x) = \int_{-\infty}^{+\infty} g(x-y)\, f(y)\, dy$$

$$\mathcal{F}(f*g) = \mathcal{F}f \cdot \mathcal{F}g$$