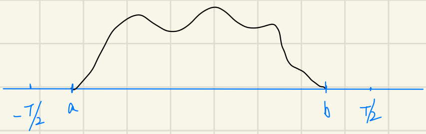
$f(t)$ has period T $(T \to +\infty)$

$$f(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{2\pi i k/T \, t}$$

$$C_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, e^{-2\pi i (k/T) t} \, dt$$

take a big T and periodize

$$C_k = \frac{1}{T} \int_a^b f(t) \, e^{-2\pi i (k/T) t} \, dt$$



$f(t) = 0$ for $t \in (-\infty, a] \cup [b, +\infty)$

$$\left| \int_a^b f(t) \, e^{-2\pi i (k/T) t} \, dt \right| \le \int_a^b |f(t)| \, |e^{-2\pi i \, k/T \, t}| \, dt$$

$$= \int_a^b |f(t)| \cdot 1 \, dt = M$$

$M/T \le C_k \le M/T$

$C_k \to 0$ as $T \to \infty$

Fix $C_k \rightsquigarrow 0$

Scale up C_k by T

$$\mathcal{F}f(k/T) = T \cdot C_k = \int_{-T/2}^{T/2} f(t) \, e^{-2\pi i (k/T) t} \, dt$$

The series is

$$f(t) = \sum_{k=-\infty}^{+\infty} \mathcal{F}f(k/T) \cdot e^{2\pi i (k/T) t} \cdot \frac{1}{T}$$

when $T \to \infty$, the discrete variable k/T tends to a continous variable $s \in (-\infty, +\infty)$

$$\mathcal{F}f(s) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-2\pi i s t} \, dt$$

$$f(t) = \int_{-\infty}^{+\infty} \mathcal{F}f(s) \cdot e^{2\pi i s t} \, ds$$

Declare Victory: Define Fourier Transform

if $f(t)$ is defined on $(-\infty, +\infty)$

the Fourier Transform and Fourier Inversion are defined by

Analysis: $$\mathcal{F}f(s) = \int_{-\infty}^{+\infty} f(t) \, e^{-2\pi i s t} \, dt$$

will need to understand the convergence of the Integral

Synthesis: $$f(t) = \int_{-\infty}^{+\infty} \mathcal{F}f(s) \, e^{2\pi i s t} \, ds$$

Fourier Transform analyzes $f(t)$ into its constituent parts $e^{2\pi i s t}$

Fourier Inversion says that we can synthesize $f(t)$ from its constituent parts

t is not always time variable, s is not always frequency variable

every signal has a spectrum, and the spectrum determines the signal

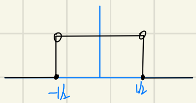
$$\mathcal{F}^{-1} g(t) = \int_{-\infty}^{+\infty} g(t)\, e^{2\pi i s t}\, ds \qquad \left(\begin{array}{l} \mathcal{F}^{-1}\mathcal{F} f = f \\ \mathcal{F}\mathcal{F}^{-1} f = f \end{array} \right)$$

$$\mathcal{F} f(0) = \int_{-\infty}^{+\infty} f(t)\, e^{-2\pi i 0 t}\, dt = \int_{-\infty}^{+\infty} f(t)\, dt$$

analogy of 0th Fourier coefficient being the average value of the function

eg. Rectangle function

$$f(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| \geq 1/2 \end{cases}$$

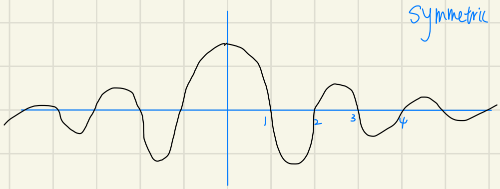


$$\mathcal{F} f(s) = \int_{-\infty}^{+\infty} f(t)\, e^{-2\pi i s t}\, dt = \int_{-1/2}^{1/2} e^{-2\pi i s t}\, dt$$

$$= \frac{1}{-2\pi i s} e^{-2\pi i s t} \Big|_{-1/2}^{1/2} = \frac{1}{\pi s} \frac{e^{\pi i s} - e^{-\pi i s}}{2i} = \frac{\sin \pi s}{\pi s}$$

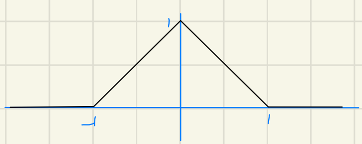
sinc function : $\operatorname{sinc}(s) = \dfrac{\sin \pi s}{\pi s}$

Symmetric



eg. triangle function

$$f(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & |t| \geq 1 \end{cases}$$



$$\mathcal{F} f(s) = \int_{-\infty}^{+\infty} f(t)\, e^{-2\pi i s t}\, dt$$

$$= \int_{-1/2}^{0} (1+t)\, e^{-2\pi i s t}\, dt + \int_{0}^{1/2} (1-t)\, e^{-2\pi i s t}\, dt$$

integrate by parts

$$= \left[(1+t) \frac{1}{-2\pi i s} e^{-2\pi i s t} \right]\Big|_{-1/2}^{0} - \int \frac{-1}{2\pi i s} e^{-2\pi i s t}\, dt + \cdots$$

$$= \frac{\sin^2 \pi s}{(\pi s)^2} = \operatorname{sinc}(s)^2$$