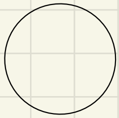
eg. heated ring



given initial temperature $f(x,t)$

$u(x,t)$ is the temperature at position x at time t

periodicity : symmetry in space $\begin{cases} f(x+1) = f(x) \\ u(x+1,t) = u(x,t) \end{cases}$

$$u(x,t) = \sum_{k=-\infty}^{+\infty} C_k(t)\, e^{2\pi i k x} \quad : \quad \text{what are } C_k(t)$$

$$\frac{du}{dt} = a \cdot \frac{d^2u}{dx^2} \qquad \text{(heat equation / diffusion equation)}$$

$$= \frac{1}{2}\frac{d^2u}{dx^2} \qquad \text{chose } a = \frac{1}{2}$$

$$\sum_{k=-\infty}^{+\infty} \frac{dC_k}{dt} e^{2\pi i k x} = \sum_{k=-\infty}^{+\infty} C_k(t) \cdot (2\pi i k)^2 e^{2\pi i k x}$$

$\Downarrow$ equate coefficient (each function has 1 F.S)

$$\frac{dC_k}{dt} = C_k(t) \cdot (2\pi i k)^2$$

$$C_k(t) = C_k(0) \cdot e^{(2\pi i k)^2 t}$$

when $t=0$

$$u(x,0) = \sum_{k=-\infty}^{+\infty} \underbrace{C_k(0)}_{\text{the kth fourier coef}} \cdot e^{2\pi i k x} \qquad \int_0^1 u(x,0)\, e^{-2\pi i k x}\, dx$$

$$u(x,t) = \sum_{k=-\infty}^{+\infty} C_k(t)\, e^{2\pi i k x}$$

$$= \sum_{k=-\infty}^{+\infty} C_k(0) \cdot e^{(2\pi i k)^2 t} \cdot e^{2\pi i k x}$$

$$= \sum_{k=-\infty}^{+\infty} \underbrace{\int_0^1 u(y,0)\, e^{-2\pi i k y}\, dy}_{\hat{f}(k)} \; e^{(2\pi i k)^2 t} \cdot e^{2\pi i k x}$$

$$= \int_0^1 \left[\sum_{k=-\infty}^{+\infty} e^{-2\pi i k y} e^{2\pi i k x} e^{-2\pi^2 k^2 t}\right] u(y,0)\, dy$$

$$= \int_0^1 \left(\sum_{k=-\infty}^{+\infty} e^{2\pi i k (x-y)} e^{-2\pi^2 k^2 t}\right) u(y,0)\, dy$$

let $g(x,t) = \sum_{k=-\infty}^{+\infty} e^{2\pi i k x} \cdot e^{-2\pi^2 k^2 t}$ $\quad f(y) = u(y,0)$

$$= \int_0^1 g(x-y,t)\, f(y)\, dy$$

convolution of initial condition $f(x)$ and the kernel $g(x,t)$

the word "kernel" is oftenly used when is appears under an integral in the context of convolution

$g(x,t)$ is called $\begin{cases} \text{heat kernel} \\ \text{fundamental sol}^{\text{n}} \text{ to heat equ}^{\text{n}} \\ \text{green's func to heat equ}^{\text{n}} \end{cases}$

When you have partial differential equations which govern many physical phenomenon the solutions often appear in the form of "convolution with a special solution (so-called fundamental solu) with the initial condition", that's sth I should expect to see

Fourier Series $\to$ Fourier Transform

periodic phenomenon $\to$ nonperiodic phenomenon

View nonperiodic function as a limiting case of the periodic function as period $\to +\infty$

The Fourier transform is the generalization/limiting case of Fourier coefficient (analysis)

The Invers Fourier transform is the generalization/limiting case of F.S (synthesis)

Set up $f(t)$ have period T, ultimately let $T \to +\infty$

building blocks $e^{2\pi i k (t/T)} = e^{2\pi i k/T \, t}$ period T

Fourier series: $$f(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{2\pi i (k/T) t}$$

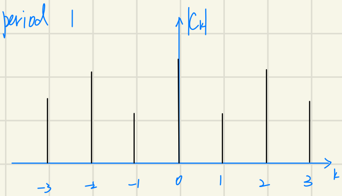
$$C_k = \frac{1}{T} \int_0^T f(t) \, e^{2\pi i (k/T) t} \, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, e^{2\pi i (k/T) t} \, dt$$

Picture of spectrum / frequency

$f(t) = \sum_{k=-\infty}^{+\infty} C_k \cdot e^{2\pi i k t}$

$f(t)$ has period 1

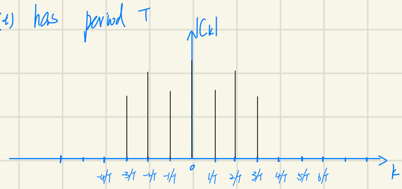


C_k's are complex, $C_k = \overline{C_k}$, $|C_k| = |C_{-k}|$

as $T \to \infty$, the spectrum become "continuous"

$f(t) = \sum_{k=-\infty}^{+\infty} C_k \cdot e^{2\pi i k/T \, t}$

$f(t)$ has period T

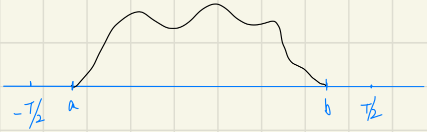


spectrum has spacing $1/T$

reciprocal relationship between time domain (period T) and frequency domain (spectrums $1/T$)

$f(t)$ has period T $(T \to +\infty)$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-2\pi i (k/T) t}\, dt$$



take a big T and periodize

$$C_k = \frac{1}{T}\int_a^b f(t)\, e^{-2\pi i (k/T) t}\, dt$$

$$\left|\int_a^b f(t)\, e^{-2\pi i (k/T) t}\, dt\right| \le \int_a^b |f(t)|\,\left|e^{-2\pi i \, k/T\, t}\right| dt$$

$$= \int_a^b |f(t)| \cdot 1 \, dt = M$$

$$M/T \le C_k \le M/T$$

$C_k \to 0$ as $T \to \infty$