Suppose $f(t)$ is square integrable $f \in L^2([0,1])$

i.e. $\int_0^1 |f(t)|^2 \, dt < \infty$ (finite energy)

if $f(t)$ periodic, period 1, $f \in L^2([0,1])$

$$\hat{f}(k) = \int_0^1 f(t) \, e^{-2\pi i k t} \, dt$$

$$\int_0^1 \left| f(t) - \sum_{k=-n}^{n} \hat{f}(k) \, e^{2\pi i k t} \right|^2 dt \to 0 \quad \text{as } n \to +\infty$$

only get these convergence results with generalization of Integral
what I learn is Riemann Integral

$$\int_0^1 e^{2\pi i m t} \cdot e^{-2\pi i n t} \, dt = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Cornerstone for introducing "geometry" into $L^2([0,1])$
Allow one to define "orthogonality" via inner product

Assume f, g are square integrable, inner product $\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} \, dt$

norm $\|f\|^2 = \int_0^1 (f(t))^2 \, dt$

f g are orthogonal if $\langle f, g \rangle = 0$

Pythagorean theorem:

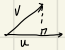
$$f \perp g \iff \|f+g\|^2 = \|f\|^2 + \|g\|^2 \iff \langle f, g \rangle = 0$$

vector $\longrightarrow$ function
Reasoning by strong analogy

$$\langle e^{2\pi i m t}, e^{2\pi i n t} \rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$e^{2\pi i m t}$'s are orthonormal

Fourier coefficients are projections of $f(t)$ on $e^{2\pi i m t}$

	vector	function
inner product	$\langle u, v \rangle = \sum_i u_i v_i$	$\langle f, g \rangle = \int_0^1 f(t) g(t) \, dt$
norm	$\|u\|_2^2 = \sum_i \|u_i\|^2$	$\|f\|^2 = \int_0^1 \|f(t)\|^2 \, dt$
orthogonality	$u \perp v \iff \langle u, v \rangle = 0$	$f \perp g \iff \langle f, g \rangle = 0$
projection	 $\langle \vec{u}, \vec{v} \rangle \cdot \vec{u}$	$\langle f, g \rangle \cdot g$

$$f(t) = \sum_{k=-\infty}^{+\infty} \hat{f}(k) \cdot e^{2\pi i k t} = \sum_{k=-\infty}^{+\infty} \langle f, e^{2\pi i k t} \rangle \cdot e^{2\pi i k t}$$

$\{ e^{2\pi i k t} \mid -\infty < k < +\infty \}$ form orthonormal basis for square integrable periodic functions

Raley's Identity:

$$\int_0^1 |f(t)|^2\,dt = \sum_{k=-\infty}^{+\infty} |\hat{f}(k)|^2$$

norm of a vector = sum of square of components

$$f(t) = \sum_{k=-\infty}^{+\infty} \hat{f}(k)\, e^{2\pi i k t}$$

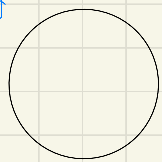
$f(t)$ has energy $\int_0^1 |f(t)|^2\,dt$ each component has energy $|\hat{f}(k)|$ ($e^{2\pi i k t}$ has energy 1)

Application to heat flow

have region in space with initial temperature distribution $f(x)$ at $t=0$ (x is space variable)

how does temperature change in position and time

eg. heated ring



given initial temperature $f(x,t)$

$u(x,t)$ is the temperature at position x at time t

periodicity: symmetry in space $\begin{cases} f(x+1) = f(x) \\ u(x+1,t) = u(x,t) \end{cases}$

$$u(x,t) = \sum_{k=-\infty}^{+\infty} C_k(t)\, e^{2\pi i k x} \quad : \text{ what are } C_k(t)$$

$$\frac{du}{dt} = a \cdot \frac{d^2u}{dx^2}$$ (heat equation / diffusion equation)

$$= \frac{1}{2}\frac{d^2u}{dx^2}$$ choose $a = \frac{1}{2}$

$$\sum_{k=-\infty}^{+\infty} \frac{dC_k}{dt} e^{2\pi i k x} = \sum_{k=-\infty}^{+\infty} C_k(t) \cdot (2\pi i k)^2 e^{2\pi i k x}$$

$\Downarrow$ equate coefficient (each function has 1 F.S)

$$\frac{dC_k}{dt} = C_k(t) \cdot (2\pi i k)^2$$

$$C_k(t) = C_0(0) \cdot e^{(2\pi i k)^2 t}$$