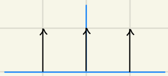
In higher dimensions, the reciprocal relationship means inverse transpose

$$\mathcal{F}(T_A f)(\xi) = \frac{1}{|A|}\,\mathcal{F}f(A^{-T}\xi)$$

· Shah function

$$Ш(x) = \sum_{p=-\infty}^{+\infty} \delta_p(x)$$



Poisson summation formula.

if ϕ is a rapidly decreasing function, then $\sum_{k=-\infty}^{+\infty} \phi(k) = \sum_{k=-\infty}^{+\infty} (\mathcal{F}\phi)(k)$

$\Phi(x) = \sum_{k=-\infty}^{+\infty} \phi(x-k)$ is periodic version of ϕ

$$\hat{\Phi}(n) = \int_0^1 \Phi(t)\, e^{-2\pi i n t}\, dt = (\mathcal{F}\phi)(n)$$

$$\begin{aligned}
\hat{\Phi}(n) &= \int_0^1 \Phi(t)\, e^{-2\pi i n t}\, dt \\
&= \int_0^1 \sum_{k=-\infty}^{+\infty} \phi(t-k)\, e^{-2\pi i n t}\, dt \\
&= \sum_{k=-\infty}^{+\infty} \int_0^1 \phi(t-k)\, e^{-2\pi i n t}\, dt \qquad u = t-k \\
&= \sum_{k=-\infty}^{+\infty} \int_{-k}^{-k+1} \phi(u)\, e^{-2\pi i n (u+k)}\, du \\
&= \sum_{k=-\infty}^{+\infty} e^{-2\pi i n k} \int_{-k}^{-k+1} \phi(u)\, e^{-2\pi i n u}\, du \\
&= \sum_{k=-\infty}^{+\infty} \int_{-k}^{-k+1} \phi(u)\, e^{-2\pi i n u}\, du \\
&= \int_{-\infty}^{+\infty} \phi(u)\, e^{-2\pi i n u}\, du = (\mathcal{F}\phi)(n)
\end{aligned}$$

$$\Phi(x) = \sum_{k=-\infty}^{+\infty} \hat{\Phi}(k)\, e^{2\pi i k x} \qquad (F.S)$$

$$= \sum_{k=-\infty}^{+\infty} (\mathcal{F}\phi)(k)\, e^{2\pi i k x}$$

$$\begin{cases} \Phi(x) = \sum_{k=-\infty}^{+\infty} \phi(x-k) \\ \Phi(x) = \sum_{k=-\infty}^{+\infty} (\mathcal{F}\phi)(k)\, e^{2\pi i k x} \end{cases}$$

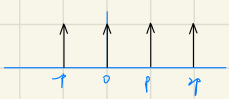
$\downarrow$ $x=0$

$$\sum_{k=-\infty}^{+\infty} \phi(k) = \sum_{k=-\infty}^{+\infty} (\mathcal{F}\phi)(k)$$

$$\langle \mathcal{F}Ш, \phi \rangle = \langle Ш, \mathcal{F}\phi \rangle = \sum_{k=-\infty}^{+\infty} \mathcal{F}\phi(k) = \sum_{k=-\infty}^{+\infty} \phi(k) = \langle Ш, \phi \rangle$$

$\mathcal{F}Ш = Ш$

$$Ш_p(x) = \sum_{k=-\infty}^{+\infty} \delta_{kp}(x)$$



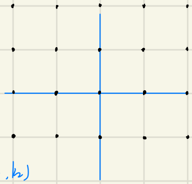
$$\begin{aligned}
Ш_p(x) &= \sum_{k=-\infty}^{+\infty} \delta(x-kp) \\
&= \sum_{k=-\infty}^{+\infty} \delta\left(p\left(\frac{x}{p}-k\right)\right) \\
&= \sum_{k=-\infty}^{+\infty} \frac{1}{p}\, \delta\left(\frac{x}{p}-k\right) \\
&= \frac{1}{p}\, Ш\left(\frac{x}{p}\right)
\end{aligned}$$

$$\begin{aligned}
(\mathcal{F}Ш_p)(s) &= \mathcal{F}\left(\tfrac{1}{p}\, Ш\left(\tfrac{x}{p}\right)\right)(s) \\
&= \tfrac{1}{p} \cdot \mathcal{F}\left(Ш\left(\tfrac{x}{p}\right)\right)(s) \\
&= \tfrac{1}{p} \cdot p \cdot \mathcal{F}Ш(ps) \\
&= Ш(ps) \\
&= \sum_{k=-\infty}^{+\infty} \delta(ps-k) = \frac{1}{p} \sum_{k=-\infty}^{+\infty} \left(s - \frac{k}{p}\right) \\
&= \frac{1}{p}\, Ш_{1/p}(s)
\end{aligned}$$

$\mathcal{F}Ш_p = \frac{1}{p}\, Ш_{1/p}$

· High Dim Shah Function

All integer coordinates form the "Integer Lattice" denoted $\mathbb{Z}^2$



$\mathbb{Z}^2$

$$Ш_{\mathbb{Z}^2}(x) = \sum_{k\in\mathbb{Z}^2} \delta(x-k) \qquad k=(k_1,k_2)$$

$$\Phi(x_1,x_2) = \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty} \phi(x_1-k_1, x_2-k_2)$$

$$\Phi(x_1,x_2) = \sum_{n_1=-\infty}^{+\infty}\sum_{n_2=-\infty}^{+\infty} \hat{\Phi}(n_1,n_2)\cdot e^{2\pi i(n_1x_1+n_2x_2)}$$

$$= \sum_{n_1=-\infty}^{+\infty}\sum_{n_2=-\infty}^{+\infty} (\mathcal{F}\phi)(n_1,n_2)\cdot e^{2\pi i(n_1x_1+n_2x_2)}$$

$$\begin{cases} \Phi(x_1,x_2) = \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty} \phi(x_1-k_1, x_2-k_2) \\ \Phi(x_1,x_2) = \sum_{n_1=-\infty}^{+\infty}\sum_{n_2=-\infty}^{+\infty} (\mathcal{F}\phi)(n_1,n_2)\, e^{2\pi i(n_1x_1+n_2x_2)} \end{cases}$$

evaluate at $x_1=0$ $x_2=0$

$$\sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty} \phi(k_1,k_2) = \sum_{n_1=-\infty}^{+\infty}\sum_{n_2=-\infty}^{+\infty} (\mathcal{F}\phi)(n_1,n_2)$$

$$\hat{\Phi}(n_1,n_2) = \int_{0}^{1}\int_{0}^{1} \Phi(x_1,x_2)\cdot e^{-2\pi i(n_1x_1+n_2x_2)}\,dx_2\,dx_1$$

$$= \int_{0}^{1}\int_{0}^{1} \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty} \phi(x_1-k_1,x_2-k_2)\, e^{-2\pi i n_1x_1}\cdot e^{-2\pi i n_2x_2}\,dx_2\,dx_1$$

$$= \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty} \int_{0}^{1}\int_{0}^{1} \phi(x_1-k_1, x_2-k_2)\, e^{-2\pi i n_1x_1}\cdot e^{-2\pi i n_2x_2}\,dx_2\,dx_1 \qquad \begin{matrix} u_1=x_1-k_1 \\ u_2=x_2-k_2\end{matrix}$$

$$= \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty} \int_{-k_1}^{-k_1+1}\int_{-k_2}^{-k_2+1} \phi(u_1,u_2)\, e^{-2\pi i n_1u_1}\cdot e^{-2\pi i n_2u_2}\cdot e^{-2\pi i n_1k_1}\, e^{-2\pi i n_2k_2}\,du_2\,du_1$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi(u_1,u_2)\, e^{-2\pi i(n_1u_1+n_2u_2)}\,du_2\,du_1$$

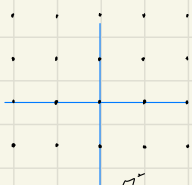
$$= (\mathcal{F}\phi)(n_1,n_2)$$

$$\langle \mathcal{F}Ш_{\mathbb{Z}^2}, \phi\rangle = \langle Ш_{\mathbb{Z}^2}, \mathcal{F}\phi\rangle$$

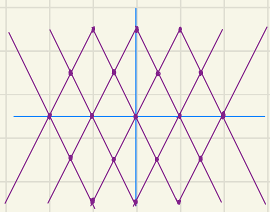
$$= \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty} \mathcal{F}\phi(k_1,k_2) = \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty} \phi(k_1,k_2)$$

$$= \langle Ш_{\mathbb{Z}^2}, \phi\rangle \qquad \mathcal{F}Ш_{\mathbb{Z}^2} = Ш_{\mathbb{Z}^2}$$

To obtain a different spacing, apply matrix multiplication to $\mathbb{Z}^2$



$\mathbb{Z}^2$



$A\mathbb{Z}^2$

$$Ш_{\mathcal{L}}(x) = \sum_{p\in\mathcal{L}} \delta(x-p)$$

$\mathcal{L}$ is the (oblique) lattice and p is any point in lattice

$$Ш_{\mathcal{L}}(x) = \sum_{p\in\mathcal{L}} \delta(x-p)$$

$$= \sum_{p\in\mathbb{Z}^2} \delta(x-Ak)$$

$$= \sum_{k\in\mathbb{Z}^2} \delta\left(A(A^{-1}x-k)\right)$$

$$= \sum_{k\in\mathbb{Z}^2} \frac{1}{|A|}\delta(A^{-1}x-k)$$

$$= \frac{1}{|A|}Ш_{\mathbb{Z}^2}(A^{-1}x)$$

$$Ш_{A\mathbb{Z}^2}(x) = \frac{1}{|A|}Ш_{\mathbb{Z}^2}(A^{-1}x)$$

$$\langle \delta(Ax), \phi\rangle = \int_{\mathbb{R}^n} \delta(Ax)\,\phi(x)\,dx \qquad u=Ax$$

$$= \int_{\mathbb{R}^n} \delta(u)\,\phi(A^{-1}u)\,d\,A^{-1}u$$

$$\delta(Ax) = \frac{1}{|A|}\delta(x) \qquad = \frac{1}{|A|}\int_{\mathbb{R}^n} \delta(u)\,\phi(A^{-1}u)\,du$$

$$= \frac{1}{|A|}\phi(A^{-1}0) = \frac{1}{|A|}\phi(0)$$

$$= \langle \frac{1}{|A|}\delta, \phi\rangle$$

$$\mathcal{F}\text{Ш}_L(\xi) = \mathcal{F}\left[\frac{1}{|A|}\text{Ш}_{\mathbb{Z}^2}(A^{-1}x)\right](\xi)$$

$$= \frac{1}{|A|}\mathcal{F}\left[\text{Ш}_{\mathbb{Z}^2}(A^{-1}x)\right](\xi)$$

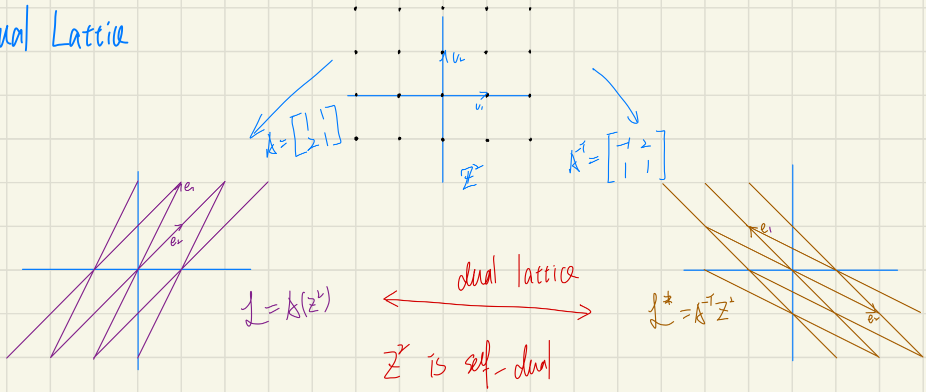
$$= \frac{1}{|A|}\frac{1}{|A^{-1}|}\mathcal{F}\text{Ш}_{\mathbb{Z}^2}(A^{T}\xi) \qquad \mathcal{F}[f(Ax)](\xi) = \frac{1}{|A|}(\mathcal{F}f)(A^{-T}\xi)$$

$$= (\mathcal{F}\text{Ш}_{\mathbb{Z}^2})(A^{T}\xi)$$

$$= \text{Ш}_{\mathbb{Z}^2}(A^{T}\xi) \qquad \mathcal{F}\text{Ш}_{\mathbb{Z}^2} = \text{Ш}_{\mathbb{Z}^2}$$

$$\mathcal{F}\text{Ш}_{A\mathbb{Z}^2}(\xi) = \text{Ш}_{\mathbb{Z}^2}(A^{T}\xi)$$

· Dual Lattice

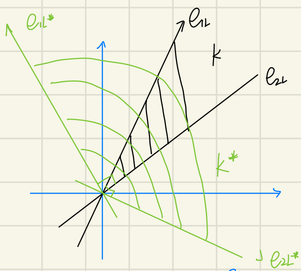


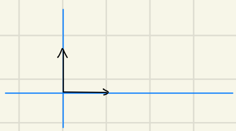
the basis of $\mathcal{L}$ are columns of A^{-T}
the basis of $\mathcal{L}^*$ are columns of A } take the pairwise inner product $(A^{-T})^T A = A^{-1}A = I$

the first base in $\mathcal{L}$ e_{1L} $\perp$ the second base in $\mathcal{L}^*$ e_{2L^*}

the second base in $\mathcal{L}$ e_{2L} $\perp$ the first base in $\mathcal{L}^*$ e_{1L^*}

A notion on dual lattice and dual cone





$\mathbb{R}^2$ is "self-dual"

the conic combination $K = \{\theta_1 e_{1L} + \theta_2 e_{2L} \mid \theta_1 \geq 0 \ \theta_2 \geq 0\}$

the conic combination $K^* = \{\theta_1 e_{1L^*} + \theta_2 e_{2L^*} \mid \theta_1 \geq 0 \ \theta_2 \geq 0\}$

K and K^* is a pair of dual cone

$K^* = \{y \mid y^T x \geq 0 \ \forall x \in K\}$

$$\mathcal{F}\text{Ш}_{A\mathbb{Z}^2}(\xi) = \text{Ш}_{\mathbb{Z}^2}(A^T\xi)$$

$$= \sum_{p\in\mathbb{Z}^2}\delta(A^T\xi - p)$$

$$= \sum_{p\in\mathbb{Z}^2}\delta(A^T(\xi - A^{-T}p))$$

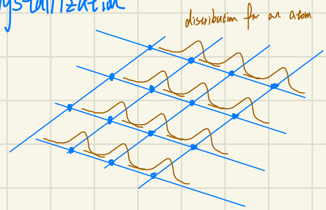
$$= \frac{1}{|A^T|}\sum_{p\in\mathbb{Z}^2}\delta(\xi - A^{-T}p)$$

$$= \frac{1}{|A|}\text{Ш}_{A^{-T}\mathbb{Z}^2}(\xi)$$

$$(\mathcal{F}\text{Ш}_L)(\xi) = \frac{1}{|A|}\text{Ш}_{L^*}(\xi)$$

$\mathcal{F}\text{Ш}_p = \frac{1}{p}\mathcal{F}\text{Ш}_{1/p}$ in 1d

· Crystallization



density for crystal

$$\rho_L(x) = (\rho * Ш_L)(x) = \sum_{p \in L} \rho(x-p)$$

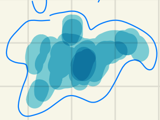
x-ray diffraction measures the Fourier Transform of ρ_L

$$\mathcal{F}\rho_L = \mathcal{F}(\rho * Ш_L) = \mathcal{F}\rho \cdot \mathcal{F}Ш_L = \mathcal{F}\rho \cdot \frac{1}{|L|}\mathcal{F}Ш_{L^*}$$

$$(\mathcal{F}\rho_L)(s) = (\mathcal{F}\rho)(s) \cdot \frac{1}{L} \cdot \sum_{p \in L^*} \delta(s-p) = \frac{1}{L} \sum_{p \in L^*} (\mathcal{F}\rho)(p) \cdot \delta_p(s)$$

· 2D F.T on Medical Imaging (Tomography)

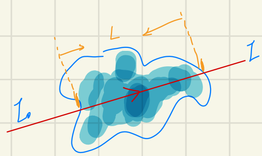
setup: 2d region (slice of body) filled with goop (bones, organs, ...)
described by density function $\mu(x_1, x_2)$



want to recover μ by x-ray measurements

approach via tomography

shoot an x-ray through the region along the line

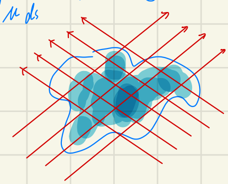


know the intensity of x ray going in I_0 and I

$I = I_0 e^{-\int_L \mu \, ds}$ the intensity drops exponentially according to the integral of μ along the line / average density

don't know μ, thus don't know $\int_L \mu \, ds$

so make lots of measurement



have measured $I = I_0 e^{-\int_L \mu \, ds}$
for different L

recover μ from $\int_L \mu \, ds$

consider the numbers $\int_L \mu \, ds$ as defining a Transform of μ depending on the line L

$(R\mu)(L) = \int_L \mu \, ds$ Radon Transform of μ

invert the Radon Transform ?