· Shift Theorem

1d: $f(t) \to \mathcal{F}f(s)$

$(\tau_b f)(t) \to e^{-2\pi i s b} \mathcal{F}f(s)$

2d: $f(x_1, x_2) \to \mathcal{F}f(\xi_1, \xi_2)$

$f(x_1-b_1, x_2-b_2) \to e^{-2\pi i (b_1\xi_1 + b_2\xi_2)}$ $\quad \tau_b f \to e^{-2\pi i b\xi} \mathcal{F}f(\xi)$

$$\mathcal{F}(\tau_{b_1,b_2} f)(\xi_1,\xi_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i (x_1\xi_1 + x_2\xi_2)} f(x_1-b_1, x_2-b_2)\, dx_1 dx_2 \qquad u_1 = x_1 - b_1 \quad u_2 = x_2 - b_2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i (u_1\xi_1 + u_2\xi_2 + b_1\xi_1 + b_2\xi_2)} f(u_1, u_2)\, du_1 du_2$$

$$= e^{-2\pi i b\xi} \cdot \mathcal{F}f(\xi_1, \xi_2)$$

· Stretch Theorem

1d: $f(x) \to \mathcal{F}f(s)$

$f(ax) \to \frac{1}{a} \mathcal{F}f\left(\frac{s}{a}\right)$

2d: $f(x_1, x_2) \to \mathcal{F}f(\xi_1, \xi_2)$

$f(a_1x_1, a_2x_2) \to \frac{1}{|a_1|}\frac{1}{|a_2|} \mathcal{F}f\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}\right)$ stretching in one domain → shrinking in the other domain still holds

$$\mathcal{F}(\tau_{a_1 a_2} f)(\xi_1, \xi_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i (\xi_1 x_1 + \xi_2 x_2)} f(a_1x_1, a_2x_2)\, dx_1 dx_2 \qquad u_1 = a_1 x_1, \; u_2 = a_2 x_2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i \left(\frac{\xi_1 x_1}{a_1} + \frac{\xi_2 x_2}{a_2}\right)} f(u_1, u_2) \frac{1}{a_1}\frac{1}{a_2}\, du_1 du_2$$

$$= \frac{1}{|a_1|}\frac{1}{|a_2|} \mathcal{F}f\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}\right) \quad (a > 0)$$

nd: general scaling: multiplication by matrix

$f(x) \to \mathcal{F}f(x)$ $\qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$ assume $\det A \neq 0$

$f(Ax) \to \frac{1}{|A|} \mathcal{F}f(A^{-T}\xi)$

$$\mathcal{F}(\tau_A f)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i (x\cdot\xi)} f(Ax)\, dx \qquad u = Ax$$

$$= \int_{\mathbb{R}^n} e^{-2\pi i (A^{-1}u\cdot\xi)} f(u) \cdot \det A^{-1}\, du \qquad \langle Bx, y\rangle = (Bx)^T y = x^T B^T y = \langle x, B^T y\rangle$$

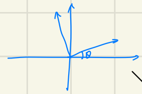
$$= \int_{\mathbb{R}^n} e^{-2\pi i (u, A^{-T}\xi)} f(u) \det A^{-1}\, du$$

$$= \frac{1}{|A|} \mathcal{F}f(A^{-T}\xi)$$

in higher dimension, the reciprocal relationship should be understood as A^{-T}

$\langle Ax, A^{-T}x\rangle = x^T A^T A^{-T} x = x^T x$

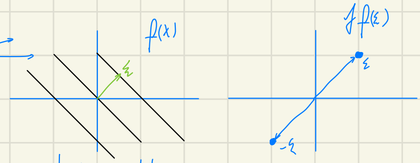
2d rotation: $A = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$



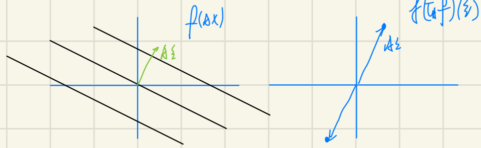
$A^T = A^{-1}$ $\quad \det A = 1$

$\mathcal{F}(\tau_A f)(\xi) = \frac{1}{|A|}\mathcal{F}f(A^{-T}\xi) = \mathcal{F}f(A\xi)$

$f(Ax) \to \mathcal{F}f(A\xi)$



↓ A is a clock-wise rotation matrix



· Delta Functions etc

$\langle \delta, \phi \rangle = \phi(\vec{0})$ $\quad \langle \delta_b, \phi \rangle = \phi(b)$

just like 1d case

$(\mathcal{F}\delta)(\xi) = 1$ $\quad \langle \mathcal{F}\delta, \phi \rangle = \langle \delta, \mathcal{F}\phi \rangle = \mathcal{F}\phi(0) = \int_{\mathbb{R}^n} e^{-2\pi i \langle 0, x \rangle} f(x)\,dx = \langle 1, \phi \rangle$

$(\mathcal{F}\delta_b)(\xi) = e^{-2\pi i b \xi}$ $\quad \langle \mathcal{F}\delta_b, \phi \rangle = \langle \delta_b, \mathcal{F}\phi \rangle = (\mathcal{F}\phi)(b) = \int_{\mathbb{R}^n} e^{-2\pi i \langle b, \xi \rangle} \phi(\xi)\,dx = \langle e^{-2\pi i b \xi}, \phi \rangle$

$(f \cdot \delta)(x) = f(0) \cdot \delta(x)$ $\quad (f \cdot \delta_b)(x) = f(b) \cdot \delta_b(x)$

1d: $\delta(ax) = \frac{1}{|a|}\delta(x)$

nd: $\delta(Ax) = \frac{1}{|A|}\delta(x)$