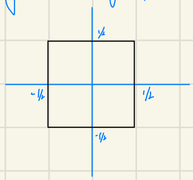
$$\mathcal{F}f(\xi) = \int_{x\in\mathbb{R}^2} e^{-2\pi i\, \xi\cdot x} f(x)\, dx$$

· Computing Higher Dimension Transform via Low-Dim F.T

separable functions

1. separable

eg 2d Rectangle function $\Pi(x_1, x_2)$



$$\Pi(x_1, x_2) = \begin{cases} 1 & |x_1| \le \frac{1}{2}\ |x_2| \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

can write $\Pi(x_1, x_2) = \Pi(x_1)\,\Pi(x_2)$

$$\mathcal{F}f(\xi_1, \xi_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i(\xi_1 x_1 + \xi_2 x_2)}\, \Pi(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{+\infty} e^{-2\pi i \xi_1 x_1}\, \Pi(x_1)\, dx_1 \cdot \int_{-\infty}^{+\infty} e^{-2\pi i \xi_2 x_2}\, \Pi(x_2)\, dx_2$$

$$= \mathcal{F}\Pi(\xi_1) \cdot \mathcal{F}\Pi(\xi_2)$$

$$= \mathrm{sinc}(\xi_1) \cdot \mathrm{sinc}(\xi_2)$$

when $f(x_1, x_2, \dots, x_n) = \tilde{f}_1(x_1) \cdot \tilde{f}_2(x_2) \cdots \tilde{f}_n(x_n)$

the $\mathcal{F}f(\xi) = \mathcal{F}\tilde{f}_1(\xi_1) \cdots \mathcal{F}\tilde{f}_n(\xi_n)$

eg. 2d Gaussian

$$G(x_1, x_2) = e^{-\pi(x_1^2 + x_2^2)}$$

$$= e^{-\pi x_1^2}\, e^{-\pi x_2^2} = G(x_1) \cdot G(x_2)$$

$$(\mathcal{F}G)(\xi_1, \xi_2) = \mathcal{F}G(\xi_1) \cdot \mathcal{F}G(\xi_2) = G(\xi_1) \cdot G(\xi_2) = e^{-\pi(\xi_1^2 + \xi_2^2)} = G(\xi)$$

$$\mathcal{F}G(\xi_1, \xi_2) = G(\xi_1, \xi_2)$$

2. radial

if polar coordinate is introduced $r = \sqrt{x_1^2 + x_2^2}$ $\theta = \arctan\frac{x_2}{x_1}$ $x_1 = r\cos\theta$ $x_2 = r\sin\theta$

$e^{-\pi(x_1^2 + x_2^2)} = e^{-\pi r^2}$ depends only on r, not x_1 and x_2 "independently"

radial functions in general depend on the distance r from some origin

$$(\mathcal{F}f)(\xi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i(x_1\xi_1 + x_2\xi_2)} f(x_1, x_2)\, dx_1\, dx_2 \qquad \begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases} \quad \begin{cases} \xi_1 = \rho\cos\phi \\ \xi_2 = \rho\sin\phi \end{cases}$$

if f is a radial function of r

$\mathcal{F}f$ is a radial function of ρ

$$= \int_{\theta=0}^{2\pi}\int_0^{+\infty} e^{-2\pi i(r\cos\theta\,\rho\cos\phi + r\sin\theta\,\rho\sin\phi)}\, \tilde{f}(r)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{+\infty} e^{-2\pi i r\rho(\cos\theta\cos\phi + \sin\theta\sin\theta)}\, \tilde{f}(r)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{+\infty} e^{-2\pi i r\rho\cos(\theta - \phi)}\, \tilde{f}(r)\, r\, dr\, d\theta$$

$$\int_{\theta=0}^{2\pi} e^{-2\pi i r\rho\cos(\theta-\phi)}\,d\theta = \int_{\theta=0}^{2\pi} e^{-2\pi i r\rho\cos\theta}\,d\theta$$ do not have a close-form expression of this integral

define $J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ia\cos\theta}\,d\theta$ (a is real)

the zeroth-order bessel function of the first kind

$$= \int_{r=0}^{+\infty}\int_{\theta=0}^{2\pi} e^{-2\pi i r\rho\cos(\theta-\phi)}\,d\theta\,\tilde{f}(r)\,r\,dr$$ circular symmetry may yields bessel functions

$$= \int_{r=0}^{+\infty} 2\pi J_0(2\pi r\rho)\,\tilde{f}(r)\,r\,dr$$

the Fourier Transform of a radial function $f(x_1, x_2) = \tilde{f}(r)$

is also a radial function $F(\rho) = \int_{r=0}^{+\infty} 2\pi J_0(2\pi r\rho)\,\tilde{f}(r)\,r\,dr$ "zeroth-order Hankel Transformation"

Hankel Transformation = Fourier Transform of a radial function in polar coordinate

- Convolution

$$(f*g)(x) = \int_{\mathbb{R}^n} f(y)\,g(x-y)\,dy$$

all formulas, properties and interpretations continue to apply

$$\mathcal{F}(f*g) = \mathcal{F}f\cdot\mathcal{F}g$$
$$\mathcal{F}(fg) = \mathcal{F}f * \mathcal{F}g$$