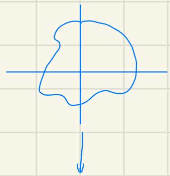
$f(x_1, x_2)$  spatial description

?? spectral description ??

analyze an image into simpler components? components are 2d complex exponentials (similar in higher dimension)

key to doing this is using vector notation

	1d	2d
spatial variable	t	$x = (x_1, x_2)$
frequency variable	s	$\xi = (\xi_1, \xi_2)$
function	$f(x)$	$f((x_1, x_2))$
product	$s \cdot t$	$\langle x, \xi \rangle = x_1\xi_1 + x_2\xi_2$
complex exponential	$e^{2\pi i s t}$	$e^{2\pi i \langle x, \xi \rangle} = e^{2\pi i (x_1\xi_1 + x_2\xi_2)}$
Fourier Transform	$\mathcal{F}f(s) = \int_{-\infty}^{+\infty} f(t)\, e^{-2\pi i s t}\, dt$	$\mathcal{F}f(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i \langle x, \xi \rangle} f(x)\, dx = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i (x_1\xi_1 + x_2\xi_2)} f(x_1, x_2)\, dx_1\, dx_2$
Inverse Fourier Transform	$\mathcal{F}^{-1}g(t) = \int_{-\infty}^{+\infty} g(s)\, e^{2\pi i s t}\, ds$	$\mathcal{F}^{-1}g(x) = \int_{\mathbb{R}^2} e^{2\pi i \langle x, \xi \rangle} g(\xi)\, d\xi$

Reciprocal relationship between the spatial domain and the frequency domain

$x = (x_1, x_2, \cdots x_n)$ is the spatial variable, imagine each x_i is a measure of length, say in "meter"

to form $\langle x, \xi \rangle = x_1\xi_1 + \cdots + x_n\xi_n$, want ξ_i to have unit $\frac{1}{\text{meter}}$

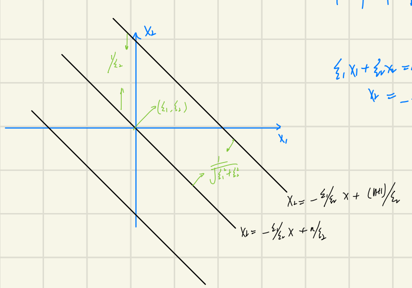
· High Dimensional Complex Exponentials

consider the 2d case

$$\exp(2\pi i\, x \cdot \xi) = \exp\left(2\pi i (x_1\xi_1 + x_2\xi_2)\right)$$

when $x \cdot \xi = n$ is an integer, $\exp(2\pi i\, x \cdot \xi) = 1$, the complex exponential has zero phase

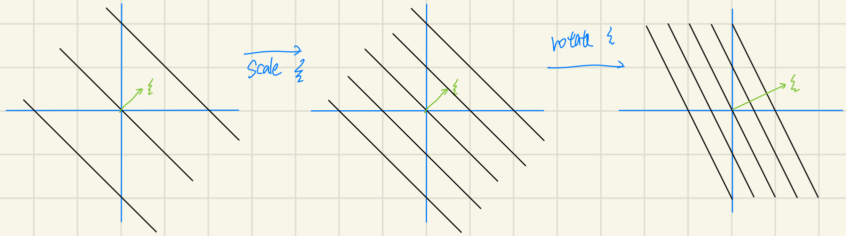
For a fixed ξ, $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ determines a family of parallel lines



$\xi_1 x_1 + \xi_2 x_2 = n$

$x_2 = -\frac{\xi_1}{\xi_2} x + \frac{n}{\xi_2}$

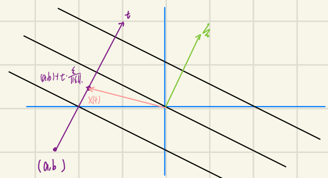
the spacing between 2 lines is the $\frac{1}{\|\xi\|_2}$. that's a reciprocal relationship



consider a point moving in the direction of ξ at a unit speed, starting at point (a,b)

the coordinate of the point at time t is

$$x(t) = (x_1(t), x_2(t)) = (a,b) + t\cdot\frac{\xi}{\|\xi\|}$$

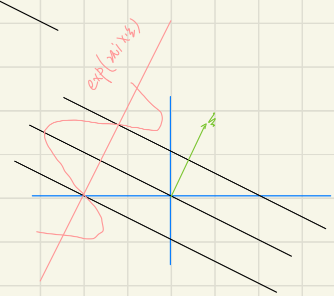


the dot product between x(t) and ξ is

$$\langle x(t), \xi\rangle = \left(a+\frac{t\cdot\xi_1}{\|\xi\|}, b+\frac{t\cdot\xi_2}{\|\xi\|}\right)\cdot(\xi_1,\xi_2)$$

$$= a\xi_1+b\xi_2 + \frac{t(\xi_1^2+\xi_2^2)}{\|\xi\|}$$

$$= a\cdot\xi_1+b\xi_2+t\cdot\|\xi\| = \langle x(0),\xi\rangle + t\cdot\|\xi\|$$



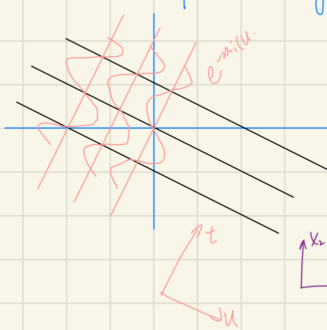
the complex exponential is

$$\exp(2\pi i\, x(t)\cdot\xi) = \exp[2\pi i(a\xi_1+b\xi_2+t\|\xi\|)]$$

$$= \underbrace{\exp(2\pi i(a\xi_1+b\xi_2))}_{\text{does not depend on } t}\cdot\underbrace{\exp(2\pi i\, t\|\xi\|)}_{\text{periodic with period } \frac{1}{\|\xi\|}}$$

$\exp(2\pi i\, x(t)\cdot\xi)$ is the frequency "along the direction of ξ"

integrating all lines that's in the direction of ξ → integrating over $\mathbb{R}^2$



just like 1d Fourier Transform

$$\int_u\int_t e^{-2\pi i\cdot t\cdot\|\xi\|}\cdot\tilde{f}(u,t)\,dt\,du$$

change coordinates

$$\iint_{x_1\,x_2} e^{-2\pi i\, x\cdot\xi}\cdot f(x_1,x_2)\,dx_2\,dx_1$$