Convolution

L is time-invariant / shift-invariant $\qquad w(x) = (Lv)(x)$

$(T_a w)(x) = (L T_a v)(x)$

a linear system given by convolution is time-invariant

$Lv = h * v$

$(Lv)(x) = \int_{-\infty}^{+\infty} h(x-y)\, v(y)\, dy$

since $(Lv)(x) = \int_{-\infty}^{+\infty} k(x,y)\, v(y)\, dy$, can conclude the impulse-response is $h(x-y)$

since $h * (T_a v) = T_a (h * v)$ the Linear system $w = Lv$ is time invariant

a time-invariant system is given by convolution

$(Lv)(x) = \int_{-\infty}^{+\infty} (L\delta_y)(x)\, v(y)\, dy$

let $(L\delta)(x) = h(x)$

then $(L\delta_y)(x) = (T_y (L\delta))(x) = (L\delta)(x-y) = h(x-y)$

$(Lv)(x) = \int_{-\infty}^{+\infty} (L\delta_y)(x)\, v(y)\, dy = \int_{-\infty}^{+\infty} (L\delta)(x-y)\, v(y)\, dy$

$= \int_{-\infty}^{+\infty} h(x-y)\, v(y)\, dy$

The impulse-response of a Linear Time-Invariant (LTI) system $h(x,y) = (L\delta)(x-y)$

same considerations hold for discrete systems.

$w = Lv$ is given by matrix multiplication

L is an LTI iff $w = h * v$ $\quad (h = L\delta \quad h[n-m] = L\delta_m)$

$w = Av$ $\quad A$ has a special form for time-invariant systems

eg.

$h = (1, 2, 3, 4)$

$w = h * v \quad (= Av \quad$ what is $A)$

$w(x) = \sum_y A\delta_y(x) \cdot v(y)$

A has columns $\begin{bmatrix} | & | & | & | \\ A\cdot\delta_0 & A\delta_1 & A\delta_2 & A\delta_3 \\ | & | & | & | \end{bmatrix}$

$\begin{bmatrix} \\ v \\ \\ \end{bmatrix}$ $\quad (1\ 2\ 3\ 4)(1 \quad 2 \quad \overset{h}{} \quad 3\ 4)$

$(\cdot \quad \cdot \quad v \quad \cdot \quad \cdot)$

since the linear system is given by convolution

$A\delta_0 = h * \delta_0 = (1, 2, 3, 4)$

$A\delta_1 = h * \delta_1 = (4, 1, 2, 3)$

$A\delta_2 = h * \delta_2 = (3, 4, 1, 2)$

$$A = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

· LTI system and Fourier Transform

time invariant system $(Lv)(x) = \int_{-\infty}^{+\infty} (L\delta_y)(x)\, v(y)\, dy = \int_{-\infty}^{+\infty} (L\delta)(x-y)\, v(y)\, dy = L\delta * v$

$w = \underbrace{h}_{\text{impulse-response}} * v \qquad W(s) = \underbrace{H(s)}_{\text{transfer function}} \cdot V(s) \qquad \left(\mathcal{F}w = (\mathcal{F}h)(\mathcal{F}v) \right)$

for LTI system, in the frequency domain, it's described by direct proportionality

$w = h * v \qquad W(s) = H(s) \cdot V(s)$

input complex exponential $v(x) = e^{2\pi i \nu x}$, what's $(Lv)(x)$?

$\mathcal{F}(e^{2\pi i \nu x}) = \delta_\nu \qquad \langle \mathcal{F}e^{2\pi i \nu x}, \phi \rangle = \langle e^{2\pi i \nu x}, \mathcal{F}\phi \rangle$

$= \int_{-\infty}^{+\infty} e^{2\pi i \nu x} (\mathcal{F}\phi)(x)\, dx$

$= (\mathcal{F}^{-1}\mathcal{F}\phi)(\nu) = \langle \delta_\nu, \phi \rangle$

$W(s) = H(s) \cdot \delta_\nu(s) = H(\nu) \cdot \delta_\nu(s)$

take the inverse F.T.

$w(x) = \mathcal{F}^{-1}(H(\nu) \cdot \delta_\nu(s)) = H(\nu) \cdot \mathcal{F}^{-1}\delta_\nu(s) = H(\nu) \cdot e^{2\pi i \nu x}$

$L(e^{2\pi i \nu x}) = H(\nu) \cdot e^{2\pi i \nu x}$

complex exponential $e^{2\pi i \nu x}$ is an eigenfunction of any LTI system, with eigenvalue $H(\nu)$ the eigenvalue $H(\nu)$ depends on the system

complex exponentials are eigenfunctions, not sin and cos

eg. $L(\cos 2\pi \nu x) = L\left(\frac{1}{2}(e^{2\pi i \nu x} + e^{-2\pi i \nu x})\right)$

$= \frac{1}{2}\left[L(e^{2\pi i \nu x}) + L(e^{-2\pi i \nu x}) \right]$

$= \frac{1}{2}\left(H(\nu) \cdot e^{2\pi i \nu x} + H(-\nu) \cdot e^{-2\pi i \nu x} \right)$

if h is real, then $\overline{H(\nu)} = H(-\nu)$

$= \frac{1}{2}\left(H(\nu) \cdot e^{2\pi i \nu x} + \overline{H(\nu)} \cdot \overline{e^{2\pi i \nu x}} \right)$

$= \mathrm{Real}\left(H(\nu)\, e^{2\pi i \nu x} \right) \qquad H(\nu) = |H(\nu)| \cdot e^{i\phi}$

$= \mathrm{Real}\left(|H(\nu)| \cdot e^{i(2\pi \nu x + \phi)} \right)$

$= |H(\nu)| \cdot \cos(2\pi \nu x + \phi)$ cos with phase shift

Discrete version

$w = Lv = h * v$

$W[m] = H[m] \cdot V[m]$

input $\quad w^k = [1 \quad e^{2\pi i \frac{k}{N}} \quad e^{2\pi i \frac{2k}{N}} \quad \cdots \quad e^{2\pi i \frac{k(N-1)}{N}}]$

$\mathcal{F}w^k = N \cdot \delta_k$

$W = H \cdot N \cdot \delta_k = H[k] \cdot N \cdot \delta_k$

$$\begin{bmatrix} | & | & | & \\ | & | & | & \cdots \\ | & | & | & \\ w^0 & w^1 & & \end{bmatrix} \begin{bmatrix} \\ w^k \\ \\ \end{bmatrix} = \begin{bmatrix} \text{—} \; w^0 \; \text{—} \\ \text{—} \; w^1 \; \text{—} \\ \vdots \\ \text{—} \; w^{N-1} \; \text{—} \end{bmatrix} \begin{bmatrix} \\ w^k \\ \\ \end{bmatrix} = \delta_k$$

(orthogonality)

take the inverse F.T

$f^{-1} W = \frac{1}{N} \cdot N \cdot H[k] \cdot W^k$

$L(W^k) = H[k] \cdot W^k$

$$\frac{1}{N} \begin{bmatrix} — W^0 — \\ — W^1 — \\ \vdots \\ — W^{N-1} — \end{bmatrix} \begin{bmatrix} N \delta_k \end{bmatrix} = \frac{1}{N} \begin{bmatrix} | & | & & | \\ W^0 & W^1 & \cdots & W^{N-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} N \cdot \delta_k \end{bmatrix} = W^k$$

$1 \; W^1 \cdots W^{N-1}$ form a orthogonal basis of eigenvectors for any LTI system

eg $w = h * v$ $\quad h = (1, 2, 3, 4)$

$$w = A v = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} v \end{bmatrix}$$

eigenvectors of the system are eigenvectors of A

$$A = \begin{bmatrix} | & | & | & | \\ W^0 & W^1 & W^2 & W^3 \\ | & | & | & | \end{bmatrix} \cdot \begin{bmatrix} H[0] & & & \\ & H[1] & & \\ & & H[2] & \\ & & & H[3] \end{bmatrix} \cdot \begin{bmatrix} — W^{-0} — \\ — W^{-1} — \\ — W^{-2} — \\ — W^{-3} — \end{bmatrix}$$

$$H = \sum_k h[k] \cdot W^{-k} = \sum_k (k+1) W^{-k} = \begin{bmatrix} 10 \\ -2+2i \\ -2 \\ -2-2i \end{bmatrix}$$