· Cascading / Composing Linear System

$$v \longrightarrow \boxed{L} \longrightarrow \boxed{M} \longrightarrow w$$

if L and M are both linear, so is the cascade $w = MLv$

when L is a linear system given by integration against a kernel

$$(Lv)(x) = \int_{-\infty}^{+\infty} k(x,y)\, v(y)\, dy$$

$$(MLv)(x) = \int_{-\infty}^{+\infty} M_x k(x,y)\, v(y)\, dy$$ (M operates on k over x)

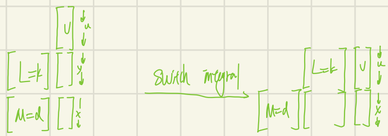
$$\int_{-\infty}^{+\infty} k(x,y)\, v(y)\, dy \approx \sum_i k(x, y_i)\, v(y_i)\, \Delta y_i$$

$$M\left[\sum_i k(x,y_i)\cdot v(y_i)\, \Delta y_i\right] = \sum_i M\big(k(x,y_i)\, v(y_i)\, \Delta y_i\big)$$

$$= \sum_i M_x\big(k(x,y_i)\big)\cdot v(y_i)\, \Delta y_i$$

$$\approx \int_{-\infty}^{+\infty} M_x\, k(x,y)\, v(y)\, dy$$

$$\left(\int_{-\infty}^{+\infty} \big(M_u\, k(u,y)\big)\Big|_x\, v(y)\, dy \right)$$



suppose M is given by integrating against a kernel d

$$(Mu)(x) = \int_{-\infty}^{+\infty} M(x,y)\, x(y)\, dy$$

$$(MLv)(x) = \int_{-\infty}^{+\infty} d(x,y)\, (Lv)(y)\, dy$$

$$= \int_{-\infty}^{+\infty}{}_{y}\, d(x,y) \int_{-\infty}^{+\infty}{}_{u}\, k(y,u)\, v(u)\, du\, dy$$

$$= \int_{-\infty}^{+\infty}{}_{u} \int_{-\infty}^{+\infty}{}_{y}\, d(x,y)\, k(y,u)\, dy\; v(u)\, du$$

$$= \int_{-\infty}^{+\infty}{}_{u} \big(M_u\, k(u,y)\big)\Big|_x\, v(y)\, dy$$

M operates on the argument x, seeing other variables as fixed

· Any Linear system is integration against a kernel

linear system M applied to kernel over u evaluated at x

$$v(x) = \int_{-\infty}^{+\infty} \delta(x-y)\, v(y)\, dy = \int_{-\infty}^{+\infty} \delta(y-x)\, v(y)\, dy$$

thinking of this

if L is a linear system, Lv is applying L to the integral

$$(Lv)(x) = L\left(\int_{-\infty}^{+\infty} \delta(u-y)\, v(y)\, dy \Big|_u \right)(x)$$

$$= \int_{-\infty}^{+\infty} \underbrace{\big(L_u \delta(u-y)\big)\Big|_x}_{\text{a kernel}}\, v(y)\, dy = \int_{-\infty}^{+\infty} k(x,y)\, v(y)\, dy$$

call $k(x,y)$ impulse-response

(how a system responds to feeding in an impulse)

Schwartz

if L is a Linear operator on distributions (with some mild assumption)

then there is a unique kernel k, with is another distribution

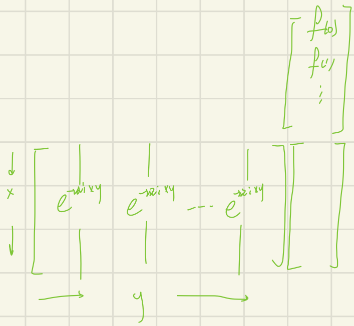
so that $Lv = \langle k, v \rangle$ (in many circumstances the pairing $\langle k, v\rangle$ is given by integration and k is L applied to δ_y)

eg. impulse-responce for Fourier Transform

$(\mathcal{F}f)(x) = \int_{-\infty}^{+\infty} e^{-2\pi i x y} f(y)\, dy$ is a linear system, integration against a kernel

$\because \langle \mathcal{F}\delta_x, \phi \rangle = \langle \delta_x, \mathcal{F}\phi \rangle = (\mathcal{F}\phi)(x) = \int_{-\infty}^{+\infty} \phi(y)\, e^{-2\pi i x y}\, dy = \langle \phi, e^{-2\pi i x y} \rangle$

$(\mathcal{F}\delta_x)(y) = e^{-2\pi i x y}$



we know: any linear system is integration against a unique kernel

$LV = \langle k, v \rangle$ and k is $L\delta_y$

Lv is defined as $(Lv)(x) = \int_{-\infty}^{+\infty} e^{-2\pi i x y}\, v(y)\, dy$

then: $k(x,y) = e^{-2\pi i x y}$ has to be $L\delta_y = \mathcal{F}\delta_y$

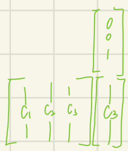
thus $e^{-2\pi i x y} = \mathcal{F}\delta_y$

once we have a Linear system written in the form of integration against a kernel

we have the impulse-response of that system

want to write linear system in form of (continuous) matrix mulitication

eg. discrete finite dimensional case

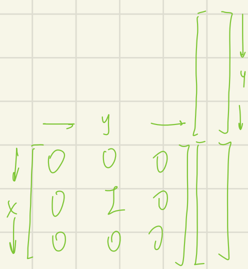


$Lv = Av$ (L is a linear system)

$(Lv)(x) = \langle A[x,:], v \rangle = \sum_y (A\delta_y)(x) \cdot v(y)$

the impulse-response is A itself

eg. switch



$LV = \Pi V$

$L\delta_y = \Pi(x)\,\delta_y(x) = \Pi(x)\,\delta(x-y) = \Pi(y)\,\delta(x-y)$

the impulse-response is $\Pi(y)\,\delta(x-y)$

$\int_{-\infty}^{+\infty} k(x,y)\, v(y)\, dy = \int_{-\infty}^{+\infty} \Pi(y)\,\delta(x-y)\, v(y)\, dy = \int_{-\infty}^{+\infty} \delta(x-y)\, \Pi(y) v(y)\, dy = \Pi(x)\, v(x) = (Lv)(x)$

· Convolution

$$Lv = h * v \qquad (Lv)(x) = \int_{-\infty}^{+\infty} h(x-y)\, v(y)\, dy$$

conclude $\quad (L\delta_y)(x) = h(x-y)$

$$h * (T_a v) = T_a (h * v)$$

$w = Lv = h * v$ $\qquad$ delay in v cause the identical delay in output

$$\begin{tikzcd}
v \arrow[r, "T_a"] \arrow[d, "L"', "h*v"] & v(t-a) \arrow[d, "h*(T_a v)"] \\
w \arrow[r, "T_a"'] & w(t-a)
\end{tikzcd}$$

the system $Lv = h * v$ is time-invariant / shift-invariant

$$w = Lv$$
$$T_a w = L\, T_a v$$

if a system is given by convolution, then time-invariant

if a system is time-invariant, it must be given by convolution

$$(Lv)(x) = \int_{-\infty}^{+\infty} (L\delta_y)(x)\, v(y)\, dy$$

let $(L\delta)(x) = h(x)$

then $\quad (L\delta_y) = L(T_y \delta) = T_y (L\delta) = T_y h(x) = h(x-y)$

$$(Lv)(x) = \int_{-\infty}^{+\infty} h(x-y)\, v(y)\, dy = \int_{-\infty}^{+\infty} (L\delta)(x-y)\, v(y)\, dy$$

switch $\quad h(x,y) = \pi(y)\, \delta(x-y)$ is not time-invariant